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Second Along With Third Hankel Determinants For Bazilevic Functions' Class

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Abstract : This research article is aimed at the target of obtaining bounds for the second as well as third Hankel determinants for the new subclass ofBazilevic function of type which is defined with the use of q-derivative operator. We have also obtained bounds for the initial coefficients of the functions belonging to the new subclass. Further we have discussed the results of theFekete-SzegÖ inequality of this subclass.

KEYWORDS : q-derivative; second as well as third Hankel determinant; Bazilevic functions;Fekete-SzegÖ problem.

Introduction

Let's assume that H denotessuch class that contains functions $\Im$ whichpossess analytic behavior in an open unit disc, $\Lambda = \{\varsigma, \forall |\varsigma| < 1\}$ and follows normalization conditions: $\Im(0) = 0,\ \Im'(0) = 1$, as well as are defined as given below:

$$\Im(\varsigma) = \varsigma + \sum_{p=2}^{\infty} d_p \varsigma^p, \quad \varsigma \in \Lambda. \qquad (1.1)$$

Let's assume that S is the subclass of H containing all such functions that are univalent in Λ. Suppose $\aleph$ be the class of analytic functions $h(\varsigma)$ in Λ which satisfy $h(0) = 1$ and $R(h(\varsigma)) > 0$ .Function $h(\varsigma) \in \aleph$ haveform:

$$h(\varsigma) = 1 + l_1\varsigma + l_2\varsigma^2 + l_3\varsigma^3 + \ldots, \varsigma \in \Lambda \qquad (1.2)$$

The Class B_β of Bazilevic functions of type β contains function $\Im \in S$ if it fulfills the following required condition:

$$R\left(\frac{\varsigma \Im'(\varsigma)\left(\Im(\varsigma)\right)^{\beta-1}}{(\psi(\varsigma))^{\beta}} \right) > 0,$$

where $\psi(\varsigma)$ is given by equation (1.1). Thomas et al. [4] introduced the class B_β. Singh et al. [5] studied the class in which $\psi(\varsigma) = \varsigma$. Let $N_q(\beta)$ be the subclass of B_β defined by,

$$N_q(\beta) = \left\{ \Im \in S; R\left(\frac{\varsigma^{1-\beta} D_q \Im(\varsigma)}{[\Im(\varsigma)]^{1-\beta}} \right) > 0 \right\}, \varsigma \in \Lambda \qquad (1.3)$$

where $0 \le \beta \le 1$, and D_q is q-derivative operator.

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Definition 1.1: The q-derivative operator is defined by (See [1],[2],[3], [29], [30], [31]):

$$D_q\Im(\varsigma)=\begin{cases}\dfrac{\Im(q\varsigma)-\Im(\varsigma)}{(q-1)\varsigma},\varsigma\neq 0\\ \Im'(\varsigma),\varsigma=0\end{cases} \quad (1.4)$$

where $q\in(0,1)$ and $\Im\in H$.

It can be noted that $\lim_{q\to 1^-} D_q\Im(\varsigma)=\Im'(\varsigma)$. Let $\Im\in H$ is given by equation (1.1), then

$$D_q\Im(\varsigma)=1+\sum_{p=2}^{\infty}[p]_q d_p\varsigma^{p-1} \quad (1.5)$$

where

$$[p]_q=\frac{1-q^p}{1-q}. \quad (1.6)$$

Thomas and Noonan [6] studied k^{th}Hankeldeterminant $H_j(p)$, for $p,j\in N$,which is defined as,

$$H_j(p)=\begin{vmatrix} d_p & d_{p+1} & \cdots & d_{p+j-1}\\ d_{p+1} & \cdots & \cdots & \vdots\\ \vdots & \vdots & \vdots & \vdots\\ d_{p+j-1} & \cdots & \cdots & d_{p+2(j-1)}\end{vmatrix} \quad (1.7)$$

For $j=2$ and $p=1$, since $d_1=1,\forall\Im\in H$ of the form of equation (1.1),

$$H_2(1)=\begin{vmatrix} d_1 & d_2\\ d_2 & d_3\end{vmatrix}=d_3-d_2^2.$$

This is the Fekete-Szegö coefficient functional $\left|d_3-\mu d_2^2\right|$ with $\mu=1$.

For $j=2$;$p=2$,

$$H_2(2)=\begin{vmatrix} d_2 & d_3\\ d_3 & d_4\end{vmatrix}=d_2d_4-d_3^2,$$

which is the second Hankel determinant. Several mathematicians like Magesh [7],Srivastava et al. [8], Ali et al. [9] and many others (see [10,11,12,13]) have considered this(second) Hankel determinant in the interest of various subgroups.For $j=3$; $p=1$, the equation (1.7) becomes,

$$H_3(1)=\begin{vmatrix} d_1 & d_2 & d_3\\ d_2 & d_3 & d_4\\ d_3 & d_4 & d_5\end{vmatrix},$$

or

$$H_3(1)=d_3(d_2d_4-d_3^2)-d_4(d_1d_4-d_2d_3)+d_5(d_1d_3-d_2^2) \quad (1.8)$$

which is called third Hankel determinant. Many authors like Babalola [14],Razaet al. [15], Maharana et al. [16], etc. worked hard to achieve this third Hankel determinant in the interest of finding different geometrical properties ofdifferent subclasses. Noor obtained Hankel determinant whose components are Bazilevic functions' coefficients. in [17] and close-to-convex functions' coefficients in [18]. The work done by Sharma [21], Srivastava [19], Magesh [20], Raina [26] and manyothers have motivated us to investigate the bound for this interesting third Hankel determinant for the subclass $N_q(\beta)$ defined by equation (1.3). In this research article, we try to obtain upper boundson $|H_3(1)|$, $|H_2(2)|$ and $\left|d_3-\mu d_2^2\right|$ for $\Im\in N_q(\beta)$. For giving the main results, the Lemmas which we used are given as follows:

Lemma 1: ([22],[23]) Suppose $h(\varsigma) \in P$ is given by equation (1.2). Then

$$\left|l_k\right| \le 2 \forall k = 1, 2... \qquad (1.9)$$

Lemma 2: [24] Suppose $h(\varsigma) \in P$ is given by equation (1.2), then

$$\left|l_2 - \mu l_1^2\right| \le \max\left\{1, |2\mu - 1|\right\}, \qquad (1.10)$$

where μ is any complex number.

Lemma 3:([24], [25])Suppose $h(\varsigma) \in P$ be given by equation (1.2), then $2l_2 = l_1^2 + y(4 - l_1^2)$,

(1.11)

and

$$4l_3 = l_1^3 + 2l_1(4-l_1^2)y - l_1(4-l_1^2)y^2 + 2(4-l_1^2)(1-|y|^2)\varsigma, \qquad (1.12)$$

for any y, ς in order that $|\varsigma| \le 1, |y| \le 1$ along with $l_1 \in [0,2]$.

Main Results

Thefirst theorem is on coefficient estimates.

Theorem 2.1: Let $\Im \in N_q(\beta)$ be given by equation (1.1), then

$$|d_2| \le \frac{2}{(\beta + q)}, \qquad (2.1)$$

$$|d_3| \le \frac{1}{(\beta + q + q^2)}, \qquad (2.2)$$

$$|d_4| \le \frac{2(\beta + 2q + 2q^2 + q^3)}{\left(\beta + q + q^2 + q^3\right)(\beta + q + q^2)(\beta + q)}, \qquad (2.3)$$

and

$$|d_5| \le \frac{\left[\begin{array}{l} 8\beta^4(q^2+q^3) + 2\beta^3(1+12q^3+12q^4) + 2\beta^2(5q+3q^2+q^3+q^4) + 2\beta(10q^2+ \\ 16q^3+13q^4+9q^5+4q^6+q^7) + 12q^3+32q^4+42q^5+38q^6+24q^7+10q^8+2q^9 \end{array}\right]}{(\beta+q)^2(\beta+q+q^2)^2(\beta+q+q^2+q^3)(\beta+q+q^2+q^3+q^4)}$$

for all $\beta \in [0,1]$ and $q \in (0,1)$.

Proof: If $\Im \in N_q(\beta)$ then from equation (1.3), we have,

$$\frac{\varsigma^{1-\beta} D_q \Im(\varsigma)}{(\Im(\varsigma))^{1-\beta}} = h(\varsigma), \qquad (2.5)$$

where $h(\varsigma)$ is given by equation (1.2) and $\Im(\varsigma)$ is given by equation (1.1).

By equating the corresponding coefficients on both sides of equation (2.5), we get:

$$d_2 = \frac{l_1}{\beta + q}, \qquad (2.6)$$

$$d_3 = \frac{l_2}{(\beta + q + q^2)} - \frac{(\beta - 1)(\beta + 2q)l_1^2}{2(\beta + q)^2(\beta + q + q^2)}, \qquad (2.7)$$

$$d_4 = \frac{l_3}{(\beta+q+q^2+q^3)} - \frac{(\beta-1)(\beta+2q+q^2)l_1 l_2}{(\beta+q)(\beta+q+q^2)(\beta+q+q^2+q^3)}$$

$$+\frac{\left[2\beta^4+\beta^3(2q^2+8q-3)+\beta^2(3q^3+6q^2-12q+1)+\beta(4q-14q^2-3q^3)+6q^2\right]l_1^3}{6(\beta+q+q^2+q^3)(\beta+q+q^2)(\beta+q)^3} \qquad (2.8)$$

and

$$d_5=\frac{l_4}{(\beta+q+q^2+q^3+q^4)}-$$

$$\frac{V_1 l_1^4}{24(\beta+q+q^2+q^3+q^4)\beta+q+q^2+q^3)(\beta+q+q^2)^2\beta+q)^4}$$

$$+\frac{V_2 l_1^2 l_2}{2(\beta+q+q^2+q^3+q^4)(\beta+q+q^2+q^3)(\beta+q+q^2)^2(\beta+q)^2}$$

$$-\frac{(\beta-1)(\beta+2q+2q^2)l_2^2}{2(\beta+q+q^2+q^3+q^4)(\beta+q+q^2)^2}-$$

$$\frac{(\beta-1)(\beta+2q+q^2+q^3)l_1 l_3}{(\beta+q+q^2+q^3+q^4)(\beta+q+q^2+q^3)(\beta+q)}$$

where

$$V_1=[6\beta^7+\beta^6(6q^3+12q^2+42q-11)+\beta^5(12q^5+37q^4+59q^3-103q^2-77q+6)+\beta^4(3q^7+41q^6+69q^5+85q^4-87q^3-205q^2+42q-1)+\beta^3(4q^8+34q^7+10q^6-49q^5-175q^4-209q^3+98q^2+17q)-\beta^2(51q^7+89q^6+150q^5-3q^4-173q^3+20q^2)+\beta(-4q^8+14q^7+38q^6+124q^5+78q^4-38q^3)-24q^4-24q^5] \tag{2.10}$$

and

$$V_2=[2\beta^5+\beta^4(-3+10q+6q^2+2q^3)+\beta^3(1-15q+8q^2+17q^3+11q^4+4q^5)$$

$$+\beta^2(5q-24q^2-20q^3+2q^4+8q^5+7q^6+q^7)+\beta(10q^2-5q^3-23q^4-18q^5-9q^6-q^7)+6q^3+10q^4+6q^5+2q^6]$$

(2.11)Taking absolute values of equation (2.6) and using Lemma 1, we get:

$$|d_2|\le\frac{2}{(\beta+q)}.$$

On taking the absolute value of equation (2.7), we obtain:

$$|d_3|=\frac{1}{(\beta+q+q^2)}\left|l_2-\frac{(\beta+2q)(\beta-1)}{2(\beta+q)^2}l_1^2\right|.$$ Now, by using lemma 2 we obtain:

$$|d_3|\le\frac{1}{(\beta+q+q^2)}.$$

By taking absolute value and applying triangle inequality in equation (2.8) we obtain:

$$|d_4|\le\frac{|l_3|}{(\beta+q+q^2+q^3)}+$$

$$\frac{(1-\beta)(\beta+2q+q^2)|l_1|}{(\beta+q+q^2+q^3)(\beta+q+q^2)(\beta+q)}\left|l_2-\frac{\begin{bmatrix}2\beta^4+\beta^3(2q^2+8q-3)+\beta^2(3q^3+6q^2\\-12q+1)+\beta(-3q^3-14q^2+4q)+6q^2\end{bmatrix}}{6(\beta+q)^2(\beta-1)(\beta+2q+q^2)}l_1^2\right|$$

.On the application of lemma 1 and 2 in the above equation we achieve the desired inequality (2.3). Similarly, from equation (2.9) we obtain:

$$|d_5|\le\frac{|l_4|}{(\beta+q+q^2+q^3+q^4)}$$

$$+\frac{V_2|l_1^2|}{2(\beta+q+q^2+q^3+q^4)(\beta+q+q^2+q^3)(\beta+q+q^2)^2(\beta+q)^2}\left|l_2-\frac{V_1 l_1^2}{12(\beta+q)^2V_2}\right|$$

$$+\frac{(1-\beta)(\beta+2q+2q^2)|l_2^2|}{2(\beta+q+q^2+q^3+q^4)(\beta+q+q^2)^2}+\frac{(1-\beta)(\beta+2q+q^2+q^3)|l_1||l_3|}{(\beta+q+q^2+q^3+q^4)(\beta+q+q^2+q^3)(\beta+q)}$$

By using lemmas 1 and 2 in above inequality we get the desired result (2.4).Hence the theorem is proved.

Theorem 2.2: Let $\Im\in N_q(\beta)$ be given by equation (1.1).Then

$$|d_2d_3-d_4|\le\frac{4[\beta^2+\beta(2q+q^2)+q^3+3q^2+2q]}{3(\beta+q+q^2)(\beta+q+q^2+q^3)}\sqrt{\frac{[\beta^2+\beta(2q+q^2)+q^3+3q^2+2q]}{X}},$$

where

$$X=4\beta^4+4\beta^3(4q+q^2)+\beta^2(-4+6q+30q^2+12q^3)+\beta(-10q+2q^2+24q^3+12q^4) \tag{2.12}$$

Proof: By using equations (2.6), (2.7) and (2.8) we obtain:

$$d_2d_3-d_4=\frac{\left[\beta^2+\beta(2q+q^2)-q+q^3\right]l_1l_2}{(\beta+q)(\beta+q+q^2)(\beta+q+q^2+q^3)}-\frac{l_3}{(\beta+q+q^2+q^3)}$$

$$+\frac{\left[\begin{matrix}-2\beta^4-\beta^3(8q+2q^2)+\beta^2(2+3q-9q^2-6q^3)+\beta(5q\\+11q^2-6q^4)+6q^3+6q^4\end{matrix}\right]l_1^3}{6(\beta+q)^3(\beta+q+q^2)(\beta+q+q^2+q^3)}$$

In the above equation, we substitute the values of l_2 and l_3 from Lemma 3. Without any loss of generality let'ssuppose that $l=l_1\in[0,2]$,thenapply triangle inequality.

$$|d_2d_3-d_4|\le|A|l^3+\frac{q(1+q)l(4-l^2)\eta}{2(\beta+q)(\beta+q+q^2)(\beta+q+q^2+q^3}$$
$$+\frac{l(4-l^2)\eta^2}{4(\beta+q+q^2+q^3)}+\frac{(4-l^2)(1-\eta^2)}{2(\beta+q+q^2+q^3)}\quad(2.13)$$
$$=M(l,\eta)$$

where $\eta=|x|$ and

$$A=\frac{[-\beta^4-\beta^3(4q+q^2)-\beta^2(-4+6q^2+3q^3)+\beta(10q+10q^2-3q^4)+6q^3+9q^4+3q^5]}{12(\beta+q)^3(\beta+q+q^2)(\beta+q+q^2+q^3)}$$

Now differentiating $M(l,\eta)$ partially with respect to η, we get:

$$\frac{\partial M}{\partial\eta}=\frac{q(1+q)l(4-l^2)}{2(\beta+q+q^2+q^3)(\beta+q+q^2)(\beta+q)}+\frac{l(4-l^2)\eta}{2(\beta+q^3+q^2+q)}$$
$$-\frac{(4-l^2)\eta}{(\beta+q^3+q^2+q)}>0.$$

Therefore M is increasing function of η.Hence,

$$\max_{0\le\eta\le1}M(l,\eta)=M(l,1)=M_1(l),\text{(2.14)}$$

where

$$M_1(l)=|A|l^3+\frac{q(1+q)l(4-l^2)}{2(\beta+q)(\beta+q+q^2)(\beta+q+q^2+q^3)}+\frac{l(4-l^2)}{4(\beta+q+q^2+q^3)}$$

Now differentiating $M_1(l)$ with respect to *l,* we obtain:

$$M_1'(l)=\frac{\beta^2+\beta(2q+q^2)+2q+3q^2+q^3}{(\beta+q)(\beta+q+q^2)(\beta+q+q^2+q^3)}-\frac{l^2X}{4(\beta+q)^3(\beta+q+q^2)(\beta+q+q^2+q^3)},$$

or

$$M_1''(l)=-\frac{lX}{2(\beta+q)(\beta+q+q^2)(\beta+q+q^2+q^3)}<0$$

,

where X is given by equation (2.12). Since $M_1''(l)<0$, by considering $M_1'(l)=0$ the maximum value of $M_1(l)$ is at

$$l=2(\beta+q)\sqrt{\frac{\beta^2+\beta(2q+q^2)+2q+3q^2+q^3}{X}}$$

Therefore,

$$\max_{0\le l\le2}M_1(l)=\frac{4[\beta^2+\beta(2q+q^2)+q^3+3q^2+2q]}{3(\beta+q+q^2)(\beta+q+q^2+q^3)}\sqrt{\frac{[\beta^2+\beta(2q+q^2)+q^3+3q^2+2q]}{X}}\quad(2.15)$$

In view of equations (2.13), (2.14) and (2.15), we obtain the asserted result.

Our next result is Fekete-Szegö problem.

Theorem 2.3: Suppose $\Im\in N_q(\beta)$ is given by equation (1.1), then

$$|d_3-\mu d_2^2|\le\left\{\begin{matrix}\frac{1}{(\beta+q^2+q)};-\frac{(\beta-1)(\beta+2q)}{2(\beta+q^2+q)}\le\mu\le\frac{\beta^2+\beta(1+2q)+2q+2q^2}{2(\beta+q^2+q)}\\\left|\frac{2\mu(\beta+q+q^2)-(\beta+2q+q^2)}{(\beta+q^2+q)(\beta+q)^2}\right|;\ Elsewhere\end{matrix}\right\}\quad(2.16)$$

Proof. From equations (2.6) and (2.7), we obtain

$$d_3-\mu d_2^2=\frac{l_2}{\beta+q+q^2}-\frac{[(\beta-1)(\beta+2q)+2\mu(\beta+q^2+q)]}{2(\beta+q^2+q)(\beta+q)^2}l_1^2$$

By taking absolute value of above equation and by the application of Lemma 3, we achieve the asserted result.

Corollary 2.4: If $\mu = 1$ in the above theorem, then we obtain:

$$\left|d_3 - d_2^2\right| \le \frac{1}{(\beta + q + q^2)}. \tag{2.17}$$

Remark 2.5: For q= 1, the corollary (2.4) becomes the same result as obtained by Kumar V. S. et al. [28].

Remark 2.6: If we assume $\beta = 1$ and $q = 1$, the corollary (2.4) becomes the same result as obtained by Haripriya M. et al. [27].

The following result is Second Hankel Determinant.

Theorem 2.7: Let $\Im$ given by equation (1.1) be in class $N_q(\beta)$; $\beta \in [0,1]$. Then we have,

$$\left|H_2(2)\right| = \left|d_2 d_4 - d_3^2\right| \le \frac{4}{(\beta + q + q^2)^2}, \tag{2.18}$$

where $q \in (0,1)$.

Proof: By using equations (2.6), (2.7) and (2.8), we have,

$$d_2 d_4 - d_3^2 = \frac{l_1 l_3}{(\beta + q)(\beta + q + q^2 + q^3)} - \frac{l_2^2}{(\beta + q + q^2)^2}$$

$$+ \frac{[\beta^2(q^3 - q^2) + \beta(q^4 - 2q^3 + q^2) + q^3 - q^4] l_1^2 l_2}{(\beta + q)^2(\beta + q + q^2)^2(\beta + q + q^2 + q^3)}$$

$$+ \frac{\begin{bmatrix} \beta^5 + \beta^4(5q + 5q^2 - 3q^3) + \beta^3(-1 + 4q^2 + 20q^3 - 8q^4) + \beta^2(-5q - 5q^2 - 9q^3 + 30q^4 \\ -6q^5) + \beta(-4q^2 - 8q^3 - 22q^4 + 18q^5) - 12q^5 \end{bmatrix} l_1^4}{12(\beta + q)^4(\beta + q + q^2)^2(\beta + q + q^2 + q^3)}$$

Now, putting values of l_2 along with l_3 from the lemma 3 in the equation (2.19) and without loss of generality assuming $l_1 = l \in [0, 2]$, we obtain:

$$\left|d_2 d_4 - d_3^2\right| = \left|W_1 l^4 + W_2 l^2(4 - l^2)y + \frac{y^2(4 - l^2)}{4}\left\{-W_3 l^2 - \frac{(4 - l^2)}{(\beta + q + q^2)^2}\right\}\right.$$

$$\left. + \frac{W_3}{2}(4 - l^2)(1 - |y|^2) l \varsigma\right|, \tag{2.20}$$

where

$$W_1 = \frac{\begin{bmatrix} \beta^5 + \beta^4(5q + 2q^2) + \beta^3(-1 + 10q^2 + 8q^3 + q^4) + \beta^2(-5q - 5q^2 + 9q^3 \\ +12q^4 + 3q^5) + \beta(-4q^2 - 8q^3 - 4q^4 + 6q^5 + 3q^6) - 6q^5 - 3q^6 \end{bmatrix}}{12(\beta + q + q^2 + q^3)(\beta + q)^4(\beta + q + q^2)^2},$$

$$W_2 = \frac{q^2}{2(\beta + q^3 + q^2 + q)(\beta + q^2 + q)^2(\beta + q)}$$

$$W_3 = \frac{1}{(\beta + q^3 + q^2 + q)(\beta + q)}.$$

On the application of triangle inequality in the equation (2.20), we get:

$$\left|d_2 d_4 - d_3^2\right| \le |W_1| l^4 + |W_2| l^2(4 - l^2)\eta + \frac{\eta^2(4 - l^2)}{4}\left|-W_3 l^2 - \frac{(4 - l^2)}{(\beta + q + q^2)^2}\right| + \frac{W_3 l(4 - l^2)}{2}\left|1 - \eta^2\right| = F(l, \eta) \tag{2.21}$$

where $\eta = |y| \in [0,1]$. Since,

$$\frac{\partial F(l,\eta)}{\partial \eta} = |W_2| l^2(4 - l^2) + \frac{\eta(4 - l^2)}{2}\left|-W_3 l^2 - \frac{(4 - l^2)}{(\beta + q + q^2)^2}\right|$$

$$-W_3 l(4 - l^2)\eta > 0$$

since $F(l,\eta)$ is the increasing function of η on $[0, 2] \times [0,1]$, therefore,

$$\max_{0 \le \eta \le 1} F(l, \eta) = F(l, 1) := M(l), \tag{2.22}$$

where

$$M(l) = |W_1| l^4 + |W_2| l^2 (4-l^2) + \frac{(4-l^2)}{4}\left| -W_3 l^2 - \frac{(4-l^2)}{(\beta+q^2+q)^2} \right|. \quad (2.23)$$

It can be observed that $M'(l) \le 0, \forall l \in [0,2]$. Therefore $M(l)$ is a function of descending order on the [0.2] such that the maximum appearsat $l=0$. Thus, the equation (2.23) gives that,

$\max_{0\le l \le 2} M(l) = M(0) = \frac{4}{(\beta+q+q^2)^2}$. Hence, from equations (2.21) and (2.22), we get,

$$|d_2 d_4 - d_3^2| \le \frac{4}{(\beta+q+q^2)^2} \quad (2.24)$$

By taking $l_1 = l = 0$ and choosing $y = -1$ in Lemma 3, we obtain $l_2 = -2$ and $l_3 = 0$. Now, by substituting these values in the equation (2.19), we can observe that the equality in attained in (2.24) which proves that the assertion is quite sharp. Hence, the proof is concluded.

In the theorem (2.7), as $q \to 1-$ some interesting result is obtained which is given below.

Corollary 2.8: If $\Im \in N_q(\beta)$ be given by equation (1.1) and we take $q=1$ then,

$$|d_2 d_4 - d_3^2| \le \frac{4}{(2+\beta)^2}.$$

Remark 2.9: If we take $\beta = 0$ in the above corollary, we obtain:

$$|d_2 d_4 - d_3^2| \le 1.$$

Our last result is third Hankel determinant.

Theorem 2.10: Let $\Im \in N_q(\beta)$ be given by equation (1.1), then

$$|H_3(1)| \le \frac{4E_1}{(\beta+q+q^2+q^3+q^4)(\beta+q+q^2+q^3)(\beta+q+q^2)^3(\beta+q)^2} + \frac{8E_2}{3(\beta+q+q^2+q^3)^2(\beta+q)(\beta+q+q^2)^2}\sqrt{\frac{\beta^2+\beta(2q+q^2)+2q+3q^2+q^3}{X}} \quad (2.25)$$

where

$$E_1 = [\beta^4 + \beta^3(4q+2q^2+2q^3+q^4) + \beta^2(6q^2+6q^3+7q^4+5q^5+2q^6+q^7) + \beta(4q^3+6q^4+8q^5+7q^6+4q^7+2q^8) + q^4 + 2q^5+3q^6+3q^7+2q^8+q^9],$$

$$E_2 = \beta^3 + \beta^2(4q+3q^2+q^3) + \beta(2q+7q^2+7q^3+4q^4+q^5) + 4q^2+10q^3+10q^4+5q^5+q^6,$$

and X is given by equation (2.12).

Proof: Take absolute value of equation (1.8) and then apply triangle inequality,

$$|H_3(1)| \le |d_2 d_4 - d_3^2||d_3| + |d_4 - d_2 d_3||d_4| + |d_3 - d_2^2||d_5| \quad (\because d_1 = 1)$$

With the use ofequation (2.17) and theorems (2.1),(2.2) and(2.7) we obtain the asserted result (2.25).

Conclusion: In this research workwe found the bounds for the second as well as third Hankel determinants for the new subclass $N_q(\beta)$ of Bazilevic function of type β. The second Hankel determinant for this subclass is bounded by $\frac{4}{(\beta+q+q^2)^2}$ and as special case if we put $q=1$ and $\beta=0$ we obtain $|d_2 d_4 - d_3^2| \le 1$.As well as theFekete-Szegö inequality of this subclass is bounded by $\frac{1}{(\beta+q^2+q)}$ or

$\left|\frac{2\mu(\beta+q+q^2)-(\beta+2q+q^2)}{(\beta+q^2+q)(\beta+q)^2}\right|$ depending on the value of μ. We havealso derived many other results including coefficient estimates and some corollaries for this subclass.

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