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Picture Fuzzy Einstein Confidence Hybrid Aggregation Operators and its Applications in MCGDM problems

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Abstract : The aim is to present novel aggregating operators named as confidence Picture fuzzy Einstein hybrid averaging and geometric operators using familiarity degree of decision makers under Picture fuzzy set (PFS) environment. Some desirable properties are also discussed. A multi criteria group decision making (MCGDM) method has been shown to solve an air quality evaluation problem. Finally, sensitivity and comparative analyses have been presented to validate the consistency of the novel aggregation operators. The results are computed, tabulated and plotted graphically.

Keywords: Picture fuzzy set, Confidence levels, Multi criteria group decision making.

Introduction

Multi criteria decision making (MCDM) is a popular technique to deal with decision-making issues in multiple-criteria scenarios. It rates various alternatives based on a variety of decision criteria before classifying them in accordance with their effectiveness. Real-world issues like waste management, choosing eco-friendly suppliers, financial investment risk, ERP system, etc. can be resolved with the aid of the MCDM technique. These real-world issues are one-dimensional, two-dimensional, or three-dimensional in nature depending on the level of uncertainty in the data due to its fuzziness or vagueness. Therefore, it is very challenging to evaluate each MCDM problem accurately using a particular tool to quantify uncertainty, an aggregation operator to aggregate data, and *t*-norm and *t*-conorm to produce arithmetic operational laws. To overcome issues associated with expressing ambiguous and uncertain opinions, the fuzzy set (FS) was introduced by Zadeh (1965). Various decision-making problems were modelled using the fuzzy sets. The need to find newer fuzzy sets capable of expressing human evaluation more accurately led to the introduction of various kinds of FS. The ordinary FS has one clear membership function and has limited use in evaluating uncertain environments.

Atanassov (1986) proposed the intuitionistic fuzzy set to present extended fuzzy preferences for decision-makers. The intuitionistic fuzzy set (IFS) has membership, non-membership, and hesitance degrees. A basic principle of the IFS is that the sum of the three degrees must equal 1 (Zu and Yagar 2006; Zu 2008). Unfortunately, the intuitionistic fuzzy set does not include the neutral degree of any element from the set. Thus, by including membership, neutral and non-membership degrees, Coung (2014) proposed PFS. PFSs have been successfully implemented in various MCDM methods (Ates and Akay 2020; Wei 2017a). Wei (2017b) studied the series of averaging and geometric aggregation operators by involving algebraic operational laws, while Garg (2017a) used Archimedean operations and proposed novel aggregation operators under PFS environment. After that, Khan et al. (2019) introduced aggregation operators based on Einstein operations under PFS environment. Jana et al. (2018) introduced novel Dombi operational laws, while Wei (2018) and Seikh and Mandal (2021) introduced novel Hamacher and frank operational laws under PFS environment, respectively. Now many researchers introduced novel aggregating operators, operational laws and MCDM problems under PFS environment (Ganie and Singh2021; Gocer 2021; Karsmti et al. 2022; Lin et al. 2021; Luo and Xing 2020; Si et al. 2021; Shahzaid et al. 2018).

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Motivation of the study

From literature survey it has been observed that the decision makers give their decision based on the performance of alternatives after their aggregation process through a suitable aggregating operator on different criteria, which is called the familiarity degree or confidence levels of the decision makers. The above discussed aggregation operators do not consider familiarity degree of decision makers under a PFS environment. However, Yu (2014) introduced the novel aggregating operators by making use of confidence level under intuitionistic fuzzy set environment. After that, Garg (2017b) proposed confidence aggregation operators under pythagorean fuzzy set environment, while Joshi and Gegov (2019) studied the series of aggregation operators with familiarity degree under q-rung orthopair fuzzy set.

Contribution of the study

From above motivation and literature survey, it has been observed that the best of knowledge no investigation has been carried out for development of any aggregation operator considering familiarity degree of decision makers with Einstein operations under PFS environment. Thus, by integrating the concept of confidence levels of decision makers and Einstein operations under PFS environment, the paper mainly proposes novel CPFEHA and CPFEHG operators, which is the main contribution of this study. The proposed study is more general, flexible and stable to solve any MCGDM problems due to the involvement of familiarity degree of decision makers under consideration.
This study is prepared as follows. Definitions of PFS, Einstein operational laws, score and accuracy functions are presented in Section 2. Section 3, develops CPFEHA and CPFEHG operators with some of their desirable properties. Based on proposed operators, a novel MCGDM approach has been designed in Section 4. After that, to exemplify the proposed MCGDM approach, a numerical assessment has been presented in Section 5. Section also provides sensitivity and comparative analyses. Finally, conclusion and future scope are provided in Section 6.

Basic Concepts

This section briefly provides basic definitions about PFS, score and accuracy functions and Einstein operations for Picture fuzzy numbers (PFNs).

PFS

Definition 1. (Cuong 2014) Let X is a universal set then PFS on X is defined as:

$$\zeta = \{x, (\mu_\zeta(x), \eta_\zeta(x), \nu_\zeta(x)) : x \in X\} \qquad (1)$$

Where, $\mu_\zeta(x),\ \eta_\zeta(x),\ \nu_\zeta(x) \in [0,1]$ are called the degrees of "positive membership", "neutral membership" and "negative membership" of x in ζ, respectively, and $\mu_\zeta(x)$, $\eta_\zeta(x)$ and $\nu_\zeta(x)$ satisfies: $0 \leq \mu_\zeta(x) + \eta_\zeta(x) + \nu_\zeta(x) \leq 1$, $\forall x \in X$. Then, $\pi_\zeta(x) = (1 - (\mu_\zeta(x) + \eta_\zeta(x) + \nu_\zeta(x))$ called the indeterminacy degree of x in ζ. Let $\beta = (\mu_\beta, \eta_\beta, \nu_\beta)$ is a Picture fuzzy number (PFN).

Score and Accuracy functions for PFN

Definition 2. (Wei 2017b) Let $\beta = (\mu_\beta, \eta_\beta, \nu_\beta)$ be a PFN then the score $(S(\beta))$ and accuracy $(H(\beta))$ functions of β are given as:

$$S(\beta) = \mu_\beta - \eta_\beta - \nu_\beta,\ H(\beta) = \mu_\beta + \eta_\beta + \nu_\beta$$

Let β_1 and β_2 be two PFNs then using score and accuracy functions, the ranking of these numbers can be done using following criterion.

(a) When $S(\beta_1) \succ S(\beta_2) \Rightarrow \beta_1 \succ \beta_2$

(b) When $S(\beta_1) = S(\beta_2)$ then

(1) When $H(\beta_1) \succ H(\beta_2) \Rightarrow \beta_1 \succ \beta_2$,
(2) When $H(\beta_1) \prec H(\beta_2) \Rightarrow \beta_1 \prec \beta_2$,
(3) When $H(\beta_1) = H(\beta_2) \Rightarrow \beta_1 = \beta_2$.

Einstein operations for PFNs

Definition 3. (Khan et al. 2019a) Let $\beta_1 = (\mu_{\beta_1}, \eta_{\beta_1}, \nu_{\beta_1})$ and $\beta_2 = (\mu_{\beta_2}, \eta_{\beta_2}, \nu_{\beta_2})$ be two PFNs, and let $\lambda > 0$. Then,

(i) $$\beta_1 \oplus \beta_2 = \left(\frac{\mu_{\beta_1} + \mu_{\beta_2}}{1 + \mu_{\beta_1}\mu_{\beta_2}}, \frac{\eta_{\beta_1}\eta_{\beta_2}}{1 + (1-\eta_{\beta_1})(1-\eta_{\beta_2})}, \frac{\nu_{\beta_1}\nu_{\beta_2}}{1 + (1-\nu_{\beta_1})(1-\nu_{\beta_2})} \right);$$

(ii) $$\beta_1 \otimes \beta_2 = \left(\frac{\mu_{\beta_1}\mu_{\beta_2}}{1 + (1-\mu_{\beta_1})(1-\mu_{\beta_2})}, \frac{\eta_{\beta_1} + \eta_{\beta_2}}{1 + \eta_{\beta_1}\eta_{\beta_2}}, \frac{\nu_{\beta_1} + \nu_{\beta_2}}{1 + \nu_{\beta_1}\nu_{\beta_2}} \right);$$

(iii) $$\beta_1^{\lambda} = \left(\frac{2\mu_{\beta_1}^{\lambda}}{(2-\mu_{\beta_1})^{\lambda} + (\mu_{\beta_1})^{\lambda}}, \frac{(1+\eta_{\beta_1})^{\lambda} - (1+\eta_{\beta_1})^{\lambda}}{(1+\eta_{\beta_1})^{\lambda} + (1+\eta_{\beta_1})^{\lambda}}, \frac{(1+\nu_{\beta_1})^{\lambda} - (1+\nu_{\beta_1})^{\lambda}}{(1+\nu_{\beta_1})^{\lambda} + (1+\nu_{\beta_1})^{\lambda}} \right);$$

(iv) $$\lambda\beta_1 = \left(\frac{(1+\mu_{\beta_1})^{\lambda} - (1+\mu_{\beta_1})^{\lambda}}{(1+\mu_{\beta_1})^{\lambda} + (1+\mu_{\beta_1})^{\lambda}}, \frac{2\eta_{\beta_1}^{\lambda}}{(2-\eta_{\beta_1})^{\lambda} + (\eta_{\beta_1})^{\lambda}}, \frac{2\nu_{\beta_1}^{\lambda}}{(2-\nu_{\beta_1})^{\lambda} + (\nu_{\beta_1})^{\lambda}} \right).$$

Aggregation operators with confidence levels under PFS environment

In this section, we proposed hybrid averaging and hybrid geometric aggregation operators using Einstein operations under confidence levels in the PFS environment.

CPFEHA operator

Definition 4. Let $\beta_j = (\mu_{\beta_j}, \eta_{\beta_j}, \nu_{\beta_j})$, where $j = 1, 2, \ldots, n$ be 'n' PFNs and $\psi_j \in [0,1]$ be the confidence level of PFNs. Then CPFEHA operator can be defined as:

$$CPFEHA\left(\langle\psi_1, \beta_1\rangle, \langle\psi_2, \beta_2\rangle, \ldots, \langle\psi_n, \beta_n\rangle\right) = \oplus_{j=1}^{n} \Omega_j (\psi_{\partial(j)} \dot{\beta}_{\partial(j)})$$
$$= \Omega_1 (\psi_{\partial(1)} \dot{\beta}_{\partial(1)}) \oplus \Omega_2 (\psi_{\partial(2)} \dot{\beta}_{\partial(2)}) \oplus \ldots \oplus \Omega_n (\psi_{\partial(n)} \dot{\beta}_{\partial(n)}) \tag{2}$$

where $\dot{\beta}_{\partial(j)}$ is j^{th} largest of the weighted Picture fuzzy values $\dot{\beta}_j (\dot{\beta}_j = n\omega_j \beta_j$, $j = 1, 2, \ldots, n)$.
Let $\Omega = (\Omega_1, \Omega_2, .., \Omega_n)^T$ and $\omega = (\omega_1, \omega_2, .., \omega_n)^T$ are the associated and weighted vectors, such that $\Omega_j, \omega_j \in [0, 1]$ and their sum is always 1.

Remark 1: If $\psi_j \in [0,1]$ then the CPFEHA operator is converted into the Picture fuzzy Einstein hybrid averaging (PFEHA) operator.

$$PFEHA(\beta_1, \beta_2, \ldots, \beta_n) = \oplus_{j=1}^{n} \Omega_j \dot{\beta}_{\partial(j)} = \Omega_1 \dot{\beta}_{\partial(1)} \oplus \Omega_2 \dot{\beta}_{\partial(2)} \oplus \ldots \oplus \Omega_n \dot{\beta}_{\partial(n)} \tag{3}$$

Theorem 1. Let $\beta_j = (\mu_{\beta_j}, \eta_{\beta_j}, \nu_{\beta_j})$, where $j = 1, 2, \ldots, n$ be 'n' PFNs and $\psi_j \in [0,1]$ be its confidence levels then the aggregated value by CPFEHA operator is also PFNs and

$$CPFEHA\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,\ldots,\langle\psi_n,\beta_n\rangle\right)$$

$$=\left(\begin{array}{l}\dfrac{\prod_{j=1}^{n}(1+\mu_{\dot\beta_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}-\prod_{j=1}^{n}(1-\mu_{\dot\beta_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(1+\mu_{\dot\beta_{\partial(j)}})^{\psi_{\delta(j)}\Omega_j}+\prod_{j=1}^{n}(1-\mu_{\dot\beta_{\delta(j)}})^{\psi_{\partial(j)}\Omega_j}},\dfrac{2\prod_{j=1}^{n}\eta_{\dot\beta_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\eta_{\dot\beta_{\delta(j)}})^{\psi_{\partial(j)}\omega_j}+\prod_{j=1}^{n}(\eta_{\dot\beta_{\delta(j)}})^{\psi_{\partial(j)}\Omega_j}},\\ \dfrac{2\prod_{j=1}^{n}\nu_{\dot\beta_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\nu_{\dot\beta_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(\nu_{\dot\beta_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}\end{array}\right) \tag{4}$$

Proof: By mathematical induction:

When $n = 2$, then

$$(\psi_1\Omega_1)\dot\beta_1=\left(\frac{(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}-(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}}{(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}+(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}},\frac{2\eta_{\dot\beta_1}^{\psi_1\Omega_1}}{(2-\eta_{\dot\beta_1})^{\psi_1\Omega_1}+(\eta_{\dot\beta_1})^{\psi_1\Omega_1}},\frac{2\nu_{\dot\beta_1}^{\psi_1\Omega_1}}{(2-\nu_{\dot\beta_1})^{\psi_1\Omega_1}+(\nu_{\dot\beta_1})^{\psi_1\Omega_1}}\right)$$

$$(\psi_2\Omega_2)\dot\beta_2=\left(\frac{(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}-(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}}{(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}+(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}},\frac{2\eta_{\dot\beta_2}^{\psi_2\Omega_2}}{(2-\eta_{\dot\beta_2})^{\psi_2\Omega_2}+(\eta_{\dot\beta_2})^{\psi_2\Omega_2}},\frac{2\nu_{\dot\beta_2}^{\psi_2\Omega_2}}{(2-\nu_{\dot\beta_2})^{\psi_2\Omega_2}+(\nu_{\dot\beta_2})^{\psi_2\Omega_2}}\right)$$

Then,

$$(\psi_1\Omega_1)\dot\beta_1\oplus(\psi_2\Omega_2)\dot\beta_2=\left(\begin{array}{c}\dfrac{\dfrac{(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}-(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}}{(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}+(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}}+\dfrac{(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}-(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}}{(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}+(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}}}{1+\left(\dfrac{(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}-(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}}{(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}+(1+\mu_{\dot\beta_1})^{\psi_1\Omega_1}}\right)\left(\dfrac{(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}-(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}}{(1+\mu_{\dot\beta_2})^{\psi_1\Omega_1}+(1+\mu_{\dot\beta_2})^{\psi_2\Omega_2}}\right)},\\ \dfrac{\left(\dfrac{2\eta_{\dot\beta_1}^{\psi_1\Omega_1}}{(2-\eta_{\dot\beta_1})^{\psi_1\Omega_1}+(\eta_{\dot\beta_1})^{\psi_1\Omega_1}}\right)\left(\dfrac{2\eta_{\dot\beta_2}^{\psi_2\Omega_2}}{(2-\eta_{\dot\beta_2})^{\psi_2\Omega_2}+(\eta_{\dot\beta_2})^{\psi_2\Omega_2}}\right)}{1+\left(1-\dfrac{2\eta_{\dot\beta_1}^{\psi_1\Omega_1}}{(2-\eta_{\dot\beta_1})^{\psi_1\Omega_1}+(\eta_{\dot\beta_1})^{\psi_1\Omega_1}}\right)\left(1-\left(\dfrac{2\eta_{\dot\beta_2}^{\psi_2\Omega_2}}{(2-\eta_{\dot\beta_2})^{\psi_2\Omega_2}+(\eta_{\dot\beta_2})^{\psi_2\Omega_2}}\right)\right)},\\ \dfrac{\left(\dfrac{2\nu_{\dot p_1}^{\psi_1\Omega_1}}{(2-\nu_{\dot p_1})^{\psi_1\Omega_1}+(\nu_{\dot p_1})^{\psi_1\Omega_1}}\right)\left(\dfrac{2\nu_{\dot\beta_2}^{\psi_2\Omega_2}}{(2-\nu_{\dot\beta_2})^{\psi_2\Omega_2}+(\nu_{\dot\beta_2})^{\psi_2\Omega_2}}\right)}{1+\left(1-\dfrac{2\nu_{\dot\beta_1}^{\psi_1\Omega_1}}{(2-\nu_{\dot\beta_1})^{\psi_1\Omega_1}+(\nu_{\dot\beta_1})^{\psi_1\Omega_1}}\right)\left(1-\left(\dfrac{2\nu_{\dot p_2}^{\psi_2\Omega_2}}{(2-\nu_{\dot\beta_2})^{\psi_2\Omega_2}+(\nu_{\dot\beta_2})^{\psi_2\Omega_2}}\right)\right)}\end{array}\right)$$

$$= \left(\begin{array}{l} \dfrac{\prod_{j=1}^{2}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} - \prod_{j=1}^{2}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{2}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\delta(j)}\Omega_j} + \prod_{j=1}^{2}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}, \dfrac{2\prod_{j=1}^{2}\eta_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{2}(2-\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} + \prod_{j=1}^{2}(\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}, \\ \dfrac{2\prod_{j=1}^{2}\nu_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{2}(2-\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} + \prod_{j=1}^{2}(\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}} \end{array} \right)$$

If Equation (4) holds for $n = k$,

$$\oplus_{j=1}^{k}(\psi_j\Omega_j)\beta_j = \left(\begin{array}{l} \dfrac{\prod_{j=1}^{k}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} - \prod_{j=1}^{k}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{k}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\delta(j)}\Omega_j} + \prod_{j=1}^{k}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}, \dfrac{2\prod_{j=1}^{k}\eta_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{k}(2-\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} + \prod_{j=1}^{k}(\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}, \\ \dfrac{2\prod_{j=1}^{k}\nu_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{k}(2-\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} + \prod_{j=1}^{k}(\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}} \end{array} \right)$$

When $n = k + 1$, then

$$\oplus_{j=1}^{k+1}(\psi_j\Omega_j)\dot{\beta}_j = \oplus_{j=1}^{k}(\psi_j\Omega_j)\dot{\beta}_j \oplus (\psi_{k+1}\Omega_{k+1})\dot{\beta}_{k+1}$$

$$= \left(\begin{array}{l} \dfrac{\prod_{j=1}^{k+1}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} - \prod_{j=1}^{k+1}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{k+1}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\delta(j)}\Omega_j} + \prod_{j=1}^{k+1}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}, \dfrac{2\prod_{j=1}^{k+1}\eta_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{k+1}(2-\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} + \prod_{j=1}^{k+1}(\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}, \\ \dfrac{2\prod_{j=1}^{k+1}\nu_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{k+1}(2-\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j} + \prod_{j=1}^{k+1}(\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}} \end{array} \right)$$

Thus, Equation (4) holds for $n = k + 1$. Then, Equation (4) holds for all n.

$$CPFEHA\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,...,\langle\psi_n,\beta_n\rangle\right)$$

$$=\left(\begin{array}{l}\dfrac{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}-\prod_{j=1}^{n}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\delta(j)}\Omega_j}+\prod_{j=1}^{n}(1-\mu_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\Omega_j}},\dfrac{2\prod_{j=1}^{n}\eta_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\eta_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\omega_j}+\prod_{j=1}^{n}(\eta_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\Omega_j}},\\ \dfrac{2\prod_{j=1}^{n}\nu_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}\end{array}\right)$$

$=\langle\psi,\ (\mu,\ \eta,\ \nu)\rangle\ \forall j$ then

Hence proof is complete.

Property 1. (Idempotency) If $\dot{\beta}_j=\dot{\beta}$

$$CPFEHA\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,...,\langle\psi_n,\beta_n\rangle\right)=\psi\dot{\beta} \tag{5}$$

Proof: $\dot{\beta}_j=\dot{\beta}=\langle\psi,\ (\mu,\ \eta,\ \nu)\rangle\quad\forall j$ and $\sum_{j=1}^{n}\Omega_j=1,$ then

$$CPFEHA\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,...,\langle\psi_n,\beta_n\rangle\right)$$

$$=\left(\begin{array}{l}\dfrac{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}-\prod_{j=1}^{n}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\delta(j)}\Omega_j}+\prod_{j=1}^{n}(1-\mu_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\Omega_j}},\dfrac{2\prod_{j=1}^{n}\eta_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\eta_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\omega_j}+\prod_{j=1}^{n}(\eta_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\Omega_j}},\\ \dfrac{2\prod_{j=1}^{n}\nu_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}\end{array}\right)$$

$$=\left(\dfrac{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}})^{\psi\Omega_j}-\prod_{j=1}^{n}(1-\mu_{\dot{\beta}})^{\psi\Omega_j}}{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}})^{\psi\Omega_j}+\prod_{j=1}^{n}(1-\mu_{\dot{\beta}})^{\psi\Omega_j}},\dfrac{2\prod_{j=1}^{n}\eta_{\dot{\beta}}^{\psi\Omega_j}}{\prod_{j=1}^{n}(2-\eta_{\dot{\beta}})^{\psi\Omega_j}+\prod_{j=1}^{n}(\eta_{\dot{\beta}})^{\psi\Omega_j}},\dfrac{2\prod_{j=1}^{n}\nu_{\dot{\beta}}^{\psi\Omega_j}}{\prod_{j=1}^{n}(2-\nu_{\dot{\beta}})^{\psi\Omega_j}+\prod_{j=1}^{n}(\nu_{\dot{\beta}})^{\psi\Omega_j}}\right)$$

$$=\left(\frac{(1+\mu_{\dot{\beta}})^{\psi\sum_{j=1}^{n}\Omega_j}-(1-\mu_{\dot{\beta}})^{\psi\sum_{j=1}^{n}\Omega_j}}{(1+\mu_{\dot{\beta}})^{\psi\sum_{j=1}^{n}\Omega_j}+(1-\mu_{\dot{\beta}})^{\psi\sum_{j=1}^{n}\Omega_j}},\frac{2\eta_{\dot{\beta}}^{\psi\sum_{j=1}^{n}\Omega_j}}{(2-\eta_{\dot{\beta}})^{\psi\sum_{j=1}^{n}\Omega_j}+\eta_{\dot{\beta}}^{\psi\sum_{j=1}^{n}\Omega_j}},\frac{2\nu_{\dot{\beta}}^{\psi\sum_{j=1}^{n}\Omega_j}}{(2-\nu_{\dot{\beta}})^{\psi\sum_{j=1}^{n}\Omega_j}+\nu_{\dot{\beta}}^{\psi\sum_{j=1}^{n}\Omega_j}}\right)$$

$$=\left(\frac{(1+\mu_{\dot{\beta}})^{\psi}-(1-\mu_{\dot{p}})^{\psi}}{(1+\mu_{\dot{\beta}})^{\psi}+(1-\mu_{\dot{p}})^{\psi}},\frac{2\eta_{\dot{\beta}}^{\psi}}{(2-\eta_{\dot{\beta}})^{\psi}+\eta_{\dot{\beta}}^{\psi}},\frac{2\nu_{\dot{\beta}}^{\psi}}{(2-\nu_{\dot{\beta}})^{\psi}+\nu_{\dot{\beta}}^{\psi}}\right)$$

$$=(\psi\dot{\beta})$$

Hence proof is complete.

Property 2. (Commutativity) Let $\beta_j=(\mu_{\beta_j},\eta_{\beta_j},\nu_{\beta_j})\left(j = 1, 2,..., n\right)$ be 'n' PFNs and ψ_j be its confidence levels. If $\beta_1^{'},\beta_2^{'},...,\beta_n^{'}$ is any permutation of $\beta_1,\beta_2,...,\beta_n$, $\beta_j^{'}=(\mu_{\beta_j}^{'},\eta_{\beta_j}^{'},\nu_{\beta_j}^{'})$ $\left(j = 1, 2,..., n\right)$, then

$$CPFEHA\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,...,\langle\psi_n,\beta_n\rangle\right)=CPFEHA\left(\langle\psi_1,\beta_1^{'}\rangle,\langle\psi_2,\beta_2^{'}\rangle,...,\langle\psi_n,\beta_n^{'}\rangle\right) \quad (6)$$

Proof: Given $\beta_1^{'},\beta_2^{'},...,\beta_n^{'}$ is any permutation of $\beta_1,\beta_2,...,\beta_n$, based on Theorem 1, it is not difficult to obtain following expression:

$$CPFEHA\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,...,\langle\psi_n,\beta_n\rangle\right)$$

$$=\left(\begin{array}{l}\frac{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}-\prod_{j=1}^{n}(1-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\delta(j)}\Omega_j}+\prod_{j=1}^{n}(1-\mu_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\Omega_j}},\frac{2\prod_{j=1}^{n}\eta_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\eta_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\omega_j}+\prod_{j=1}^{n}(\eta_{\dot{\beta}_{\delta(j)}})^{\psi_{\partial(j)}\Omega_j}},\\ \frac{2\prod_{j=1}^{n}\nu_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}\end{array}\right)$$

$$=\left(\begin{array}{l}\frac{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}_{\partial(j)}^{'}})^{\psi_{\partial(j)}\Omega_j}-\prod_{j=1}^{n}(1-\mu_{\dot{\beta}_{\partial(j)}^{'}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(1+\mu_{\dot{\beta}_{\partial(j)}^{'}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(1-\mu_{\dot{\beta}_{\partial(j)}^{'}})^{\psi_{\partial(j)}\Omega_j}},\frac{2\prod_{j=1}^{n}\eta_{\dot{\beta}_{\partial(j)}^{'}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\eta_{\dot{\beta}_{\partial(j)}^{'}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(\eta_{\dot{\beta}_{\partial(j)}^{'}})^{\psi_{\partial(j)}\Omega_j}},\\ \frac{2\prod_{j=1}^{n}\nu_{\dot{p}_{\partial(j)}^{'}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\nu_{\dot{\beta}_{\partial(j)}^{'}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(\nu_{\dot{\beta}_{\partial(j)}^{'}})^{\psi_{\partial(j)}\Omega_j}}\end{array}\right)$$

$$=CPFEHA\left(\langle\psi_1,\beta_1^{'}\rangle,\langle\psi_2,\beta_2^{'}\rangle,...,\langle\psi_n,\beta_n^{'}\rangle\right)$$

Hence proof is complete.

Property 3. (Boundedness) Let

$\dot{\beta}^{-} = \left(\min_j\left\{\psi_j,\mu_j\right\},\ \max_j\left\{\psi_j,\eta_j\right\},\ \max_j\{\psi_j,\nu_j\}\right)$ and $\dot{\beta}^{+} = \left(\max_j\left\{\psi_j,\mu_j\right\},\ \min_j\left\{\psi_j,\eta_j\right\},\ \min_j\{\psi_j,\nu_j\}\right)$ then

$$\dot{\beta}^{-} \le CPFEHA\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,...,\langle\psi_n,\beta_n\rangle\right) \le \dot{\beta}^{+} \tag{7}$$

Proof: The proof is easy so it is omitted here.

Property 4. (Monotonicity) If β_j and $\beta_j^{'}$ be two different collections of PFNs such that $\beta_j \le \beta_j^{'}$ for all j then

$$CPFEHA\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,...,\langle\psi_n,\beta_n\rangle\right) \le CPFEHA\left(\langle\psi_1,\beta_1^{'}\rangle,\langle\psi_2,\beta_2^{'}\rangle,...,\langle\psi_n,\beta_n^{'}\rangle\right) \tag{8}$$

Proof: The proof is easy so it is omitted here.

3.2. *CPFEHG operator*

Definition 5. Let $\beta_j = (\mu_{\beta_j},\eta_{\beta_j},\nu_{\beta_j})$ $(j = 1, 2,..., n)$ be a collection of PFNs and $\psi_j \in [0,1]$ be the confidence levels of PFNs. Then CPFEHG operator can be defined as:

$$CPFEHG\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,...,\langle\psi_n,\beta_n\rangle\right) = (\dot{\beta}_{\partial(1)}^{\psi_{\partial(1)}})^{\Omega_1} \otimes (\dot{\beta}_{\partial(2)}^{\psi_{\partial(2)}})^{\Omega_2} \otimes ... \otimes (\dot{\beta}_{\partial(n)}^{\psi_{\partial(n)}})^{\Omega_n} \tag{9}$$

Where $\dot{\beta}_{\partial(j)}$ is the j^{th} largest of the weighted Picture fuzzy values $\dot{\beta}_j (\dot{\beta}_j = (\beta_j)^{n\omega_j}, j = 1, 2, ..., n)$, $\Omega = (\Omega_1,\Omega_2,..,\Omega_n)^T$ is the weighted vector of the CPFEHG operator, such that $\Omega_j \in [0, 1]$ and $\sum_{j=1}^{n}\Omega_j = 1$. Let $\omega = (\omega_1,\omega_2,..,\omega_n)^T$ be the weight vector of these PFNs such that $\omega_j \in [0, 1]$, $\sum_{j=1}^{n}\omega_j = 1$ and n is the balancing coefficient, which plays a role of balance.

Remark 2: If $l_1 = l_2 = \ldots = l_n = 1$ then the CPFEHG operator is converted into the Picture fuzzy Einstein hybrid geometric (PFEHG) operator.

$$PFEHG(\beta_1,\beta_2,...,\beta_n) = \otimes_{j=1}^{n}\dot{\beta}_{\partial(j)}^{\Omega_j} = \dot{\beta}_{\partial(1)}^{\Omega_1} \otimes \dot{\beta}_{\partial(2)}^{\Omega_2} \otimes ... \otimes \dot{\beta}_{\partial(3)}^{\Omega_3} \tag{10}$$

According to the operational laws for PFNs, the following theorem can be obtained:

Theorem 2. Let $\beta_j = (\mu_{\beta_j},\eta_{\beta_j},\nu_{\beta_j})(j = 1, 2,..., n)$ be 'n' PFNs and $\psi_j \in [0,1]$ be its confidence levels then the aggregated value by CPFEHG operator is also PFNs and

$$CPFEHG\left(\langle\psi_1,\beta_1\rangle,\langle\psi_2,\beta_2\rangle,\ldots,\langle\psi_n,\beta_n\rangle\right)$$

$$= \left(\begin{array}{l} \dfrac{2\prod_{j=1}^{n}\mu_{\dot{\beta}_{\partial(j)}}^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(2-\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(\mu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}, \dfrac{\prod_{j=1}^{n}(1+\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}-\prod_{j=1}^{n}(1-\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(1+\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(1-\eta_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}, \\ \dfrac{\prod_{j=1}^{n}(1+\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}-\prod_{j=1}^{n}(1-\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}}{\prod_{j=1}^{n}(1+\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}+\prod_{j=1}^{n}(1-\nu_{\dot{\beta}_{\partial(j)}})^{\psi_{\partial(j)}\Omega_j}} \end{array} \right) \quad (11)$$

Proof: Same as above Theorem.

MCGDM approach with confidence levels

Consider MCGDM problem with a combination of n different alternatives $B = \{B_1, B_2, \ldots, B_n\}$ and m criteria $D = \{D_1, D_2, \ldots, D_m\}$ whose weight vector is $\Omega = (\Omega_1, \Omega_2, .., \Omega_n)^T$ satisfying $\Omega_j \in [0,1]$ and $\sum_{j=1}^{n}\Omega_j = 1$. Let there are r set of experts denoted by $A = \{A_1, A_2, \ldots, A_r\}$ with weight vector $\tau = (\tau_1, \tau_2, \ldots, \tau_r)^T$ satisfying $\tau_s > 0$, $s = 1,2,\ldots,r$ and $\sum_{s=1}^{r}\tau_s = 1$ which are evaluating each B_i with D_j in the form of PFNs. The steps to implement the proposed MCGDM method for evaluating the best alternative are as follows.

Step 1 For each decision maker A_r, collect the information about each B_i under the D_j and represent it in the form of PFNs $C^s = \langle \psi^{s}_{ij}, (\mu^s_{ij}, \eta^s_{ij}, \nu^s_{ij})\rangle_{n\times m}$ for $i = 1,2,\ldots,n$; $j = 1,2,\ldots,m$ and $s = 1,2,\ldots,r$ as

$$C^s_{n\times m} = \begin{pmatrix} \langle \psi^{s}_{11}, (\mu^s_{11}, \eta^s_{11}, \nu^s_{11})\rangle & \langle \psi^{s}_{12}, (\mu^s_{12}, \eta^s_{12}, \nu^s_{12})\rangle & \cdots & \langle \psi^{s}_{1m}, (\mu^s_{1m}, \eta^s_{1m}, \nu^s_{1m})\rangle \\ \langle \psi^{s}_{21}, (\mu^s_{21}, \eta^s_{21}, \nu^s_{21})\rangle & \langle \psi^{s}_{22}, (\mu^s_{22}, \eta^s_{22}, \nu^s_{22})\rangle & \cdots & \langle \psi^{s}_{2m}, (\mu^s_{2m}, \eta^s_{2m}, \nu^s_{2m})\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \psi^{s}_{n1}, (\mu^s_{n1}, \eta^s_{n1}, \nu^s_{n1})\rangle & \langle \psi^{s}_{n2}, (\mu^s_{n2}, \eta^s_{n2}, \nu^s_{n2})\rangle & & \langle \psi^{s}_{nm}, (\mu^s_{nm}, \eta^s_{nm}, \nu^s_{nm})\rangle \end{pmatrix}$$

Where ψ^{s}_{ij}, $(0 \le \psi^{s}_{ij} \le 1)$ denotes the confidence level of the experts.

Step 2 The following transformation is used to normalize distinct types of criteria.

$$Q = [q_{ij}] = \begin{cases} \langle \psi_{ij}, (\mu_{ij}, \eta_{ij}, \nu_{ij})\rangle, & \text{for benifit criteria } D_j, \\ \langle \psi_{ij}, (\nu_{ij}, \eta_{ij}, \mu_{ij})\rangle, & \text{for cost criteria } D_j \end{cases}$$

Step 3 Aggregate all r decision matrices $C^s, s = 1, 2, \ldots, r$ as provided by r experts into a collective decision matrix by employing proposed CPFEHA operator

$$p_{ij} = CPFEHA(q_{ij}^1, q_{ij}^2, \ldots, q_{ij}^r)$$

$$= \left(\frac{\prod_{r=1}^{s}\left(1+\mu_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r} - \prod_{r=1}^{s}\left(1-\mu_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r}}{\prod_{r=1}^{s}\left(1+\mu_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r} + \prod_{r=1}^{s}\left(1-\mu_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r}}, 2\frac{\prod_{r=1}^{s}\left(\eta_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r}}{\prod_{r=1}^{s}\left(2-\eta_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r} + \prod_{r=1}^{s}\left(\eta_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r}}, \right.$$
$$\left. 2\frac{\prod_{r=1}^{s}\left(\nu_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r}}{\prod_{r=1}^{s}\left(2-\nu_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r} + \prod_{r=1}^{s}\left(\nu_{\dot{\beta}_{\partial(ij)}}\right)^{\psi_{\partial(ij)}^{r}\tau_r}} \right)$$

Step 4 Aggregate PFNs β_{ij} by using $PFEHA$ operator

$$\beta_i = PFEHA(\beta_{i1}, \beta_{i2}, \ldots, \beta_{in})$$
$$= \left(\frac{\prod_{j=1}^{n}\left(1+\mu_{ij}\right)^{\Omega_j} - \prod_{j=1}^{n}\left(1-\mu_{ij}\right)^{\Omega_j}}{\prod_{j=1}^{n}\left(1+\mu_{ij}\right)^{\Omega_j} + \prod_{j=1}^{n}\left(1-\mu_{ij}\right)^{\Omega_j}}, 2\frac{\prod_{j=1}^{n}\left(\eta_{ij}\right)^{\Omega_j}}{\prod_{j=1}^{n}\left(2-\eta_{ij}\right)^{\Omega_j} + \prod_{j=1}^{n}\left(\eta_{ij}\right)^{\Omega_j}}, 2\frac{\prod_{j=1}^{n}\left(\nu_{ij}\right)^{\Omega_j}}{\prod_{j=1}^{n}\left(2-\nu_{ij}\right)^{\Omega_j} + \prod_{j=1}^{n}\left(\nu_{ij}\right)^{\Omega_j}} \right)$$

Step 5 Evaluate the score values $S(\beta_i)$ for each $B_i (i = 1,2,\ldots,n)$.

Step 6 Finally, rank the all alternatives and best alternative is then select.

Illustrative example

To demonstrate the proposed method, the air quality evaluation (adapted from Shahzaib et al. (2018)) was solved using newly developed novel aggregating operators. Three stations (C^1, C^2, C^3) that use a weighting vector of $\tau = (0.314, 0.355, 0.331)^T$ to evaluate air quality can be considered decision makers. The weighted vector of three measured indexes is $\Omega = (0.40, 0.20, 0.40)^T$, and they are denoted as:

1. $SO_2(D_1)$,
2. $NO_2(D_2)$ and
3. $PM_{10}(D_3)$.

The values obtained from stations describe in Tables 1, 2 and 3.

5.1. Procedural steps for group decision making

Step 1 The assessment matrix $Q^s = \langle \psi_{ij}^{s}, (\mu_{ij}^s, \eta_{ij}^s, \nu_{ij}^s) \rangle$, where s=1, 2, 3, whose weight vector is $\tau = (0.314, 0.355, 0.331)^T$ has been collected and presented in Tables 1, 2 and 3.

Step 2 The criteria are all of the benefit type, so no need to change it.

Table 1: Assessment values of expert C^1

Alternative	D_1	D_2	D_3
B_1	$\langle 0.90, (0.265, 0.150, 0.385) \rangle$	$\langle 0.90, (0.330, 0.190, 0.280) \rangle$	$\langle 0.90, (0.245, 0.175, 0.380) \rangle$
B_2	$\langle 0.90, (0.345, 0.145, 0.310) \rangle$	$\langle 0.90, (0.430, 0.190, 0.180) \rangle$	$\langle 0.90, (0.245, 0.275, 0.310) \rangle$
B_3	$\langle 0.90, (0.365, 0.150, 0.335) \rangle$	$\langle 0.90, (0.480, 0.310, 0.105) \rangle$	$\langle 0.90, (0.340, 0.310, 0.190) \rangle$
B_4	$\langle 0.90, (0.430, 0.210, 0.170) \rangle$	$\langle 0.90, (0.460, 0.145, 0.165) \rangle$	$\langle 0.90, (0.310, 0.430, 0.070) \rangle$

Table 2: Assessment values of expert C^2

Alternative	D_1	D_2	D_3
B_1	$\langle 0.80,(0.125,0.470,0.200)\rangle$	$\langle 0.80,(0.220,0.400,0.160)\rangle$	$\langle 0.80,(0.345,0.410,0.125)\rangle$
B_2	$\langle 0.80,(0.355,0.335,0.120)\rangle$	$\langle 0.80,(0.300,0.170,0.330)\rangle$	$\langle 0.80,(0.205,0.430,0.105)\rangle$
B_3	$\langle 0.80,(0.315,0.380,0.100)\rangle$	$\langle 0.80,(0.340,0.265,0.195)\rangle$	$\langle 0.80,(0.280,0.520,0.190)\rangle$
B_4	$\langle 0.80,(0.365,0.375,0.135)\rangle$	$\langle 0.80,(0.355,0.220,0.305)\rangle$	$\langle 0.80,(0.325,0.405,0.090)\rangle$

Table 3: Assessment values of expert C^3

Alternative	D_1	D_2	D_3
B_1	$\langle 0.70,(0.260,0.075,0.395)\rangle$	$\langle 0.70,(0.220,0.414,0.160)\rangle$	$\langle 0.70,(0.255,0.370,0.275)\rangle$
B_2	$\langle 0.70,(0.270,0.160,0.360)\rangle$	$\langle 0.70,(0.320,0.015,0.465)\rangle$	$\langle 0.70,(0.135,0.575,0.090)\rangle$
B_3	$\langle 0.70,(0.245,0.365,0.290)\rangle$	$\langle 0.70,(0.250,0.570,0.110)\rangle$	$\langle 0.70,(0.175,0.330,0.165)\rangle$
B_4	$\langle 0.70,(0.390,0.340,0.160)\rangle$	$\langle 0.70,(0.305,0.435,0.120)\rangle$	$\langle 0.70,(0.465,0.425,0.076)\rangle$

Step 3 In order to combine decision matrices C^s, $s = 1,2,3$ to form a collective decision matrix use CPFEHA operator, where $\tau = (0.314,0.355,0.331)^T$ (see Table 4).

Table 4: Comprehensive decision matrix *C* using CPFEHA operator

Alternative	D_1	D_2	D_3
B_1	$(0.1723,0.2844,0.4104)$	$(0.2091,0.4144,0.2905)$	$(0.2291,0.3985,0.3348)$
B_2	$(0.2648,0.2982,0.3326)$	$(0.2865,0.1598,0.3985)$	$(0.1598,0.5006,0.2366)$
B_3	$(0.2531,0.3741,0.3088)$	$(0.2982,0.4538,0.2166)$	$(0.2191,0.4802,0.2752)$
B_4	$(0.3222,0.3990,0.2420)$	$(0.3085,0.3330,0.2824)$	$(0.2941,0.5151,0.1444)$

Step 4 Use PFEHA operator to aggregate all preference values, where $\Omega = (0.40,0.20,0.40)^T$ (see Table 5).

Table 5: The overall preference value computed from CPFEHA operator

Alternative	$CPFEHA$
B_1	$(0.2098,0.3792,0.3300)$
B_2	$(0.2530,0.2617,0.3351)$
B_3	$(0.2646,0.4255,0.2623)$
B_4	$(0.3055,0.4132,0.2105)$

Step 5 Since $S(B_1) - 0.1201$, $S(B_2) = -0.0821$, $S(B_3) = 0.0022$, and $S(B_4) = 0.0950$.

Step 6 We get $S(B_4) > S(B_3) > S(B_2) > S(B_1)$. Thus, B_4 is best.

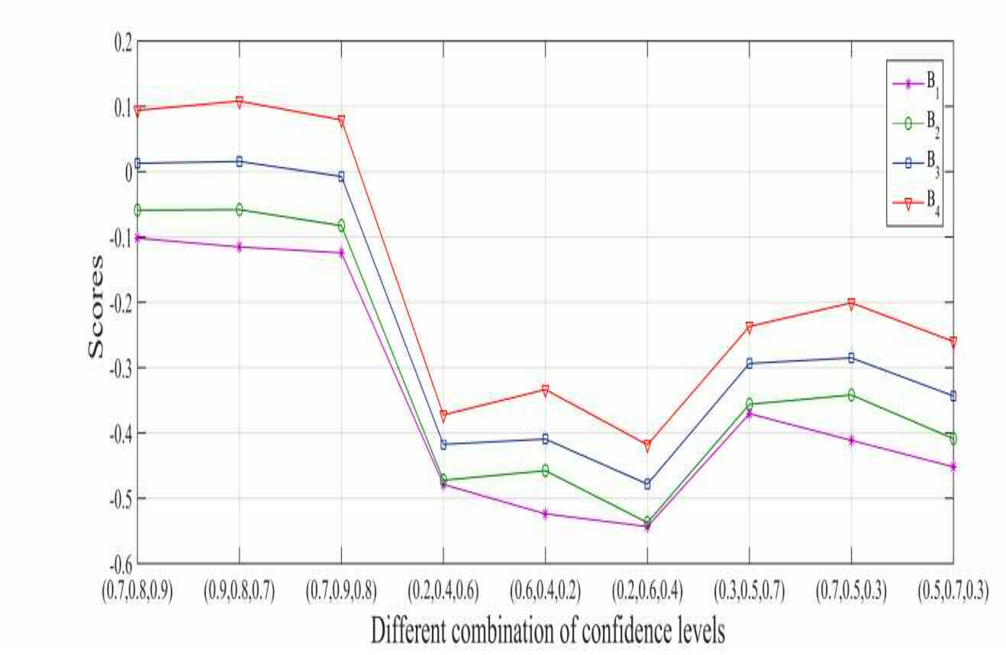
Sensitivity Analysis

The effect of different combinations of confidence levels $\psi = (\psi_{ij}^1, \psi_{ij}^2, \psi_{ij}^3)$; $i = 1,2,3,4$; $j = 1,2,3$ on final decision making is used to solve the current MCGDM problem is examined in this section. The computed results are tabulated in Table 6 and plotted in Fig. 1 using all of the considered combinations. Alternatives have different score values for different combinations of ψ, as shown in Table 6 and Fig. 1, but their ranking order is the same. For the CPFEHA operator, the best and worst alternatives are B_4 and B_1, respectively, from all the selected combinations. As a result, we can conclude that the CPFEHA operator is reliable and consistent across confidence levels.

Table 6: Ranking results obtained by utilizing different combinations of confidence level *l* in CPFEHA operator

Confidence Level $\psi = (\psi_{ij}^1, \psi_{ij}^2, \psi_{ij}^3)$	Score values	Ranking Results	Best alternative
ψ = (0.7,0.8,0.9)	$S(B_1) = -0.1019, S(B_2) = -0.0589, S(B_3) = 0.0128, S(B_4) = 0.0939$	$B_4 > B_3 > B_2 > B_1$	B_4
ψ = (0.9,0.8,0.7)	$S(B_1) = -0.1153, S(B_2) = -0.0583, S(B_3) = 0.0156, S(B_4) = 0.1080$	$B_4 > B_3 > B_2 > B_1$	B_4
ψ = (0.7,0.9,0.8)	$S(B_1) = -0.1242, S(B_2) = -0.0826, S(B_3) = -0.0074, S(B_4) = 0.0793$	$B_4 > B_3 > B_2 > B_1$	B_4
ψ = (0.2,0.4,0.6)	$S(B_1) = -0.4789, S(B_2) = -0.4725, S(B_3) = -0.4175, S(B_4) = -0.3724$	$B_4 > B_3 > B_2 > B_1$	B_4
ψ = (0.6,0.4,0.2)	$S(B_1) = -0.5239, S(B_2) = -0.4578, S(B_3) = -0.4094, S(B_4) = -0.3336$	$B_4 > B_3 > B_2 > B_1$	B_4
ψ = (0.2,0.6,0.4)	$S(B_1) = -0.5433, S(B_2) = -0.5372, S(B_3) = -0.4783, S(B_4) = -0.4182$	$B_4 > B_3 > B_2 > B_1$	B_4
ψ = (0.3,0.5,0.7)	$S(B_1) = -0.3706, S(B_2) = -0.3562, S(B_3) = -0.2936, S(B_4) = -0.2370$	$B_4 > B_3 > B_2 > B_1$	B_4
ψ = (0.7,0.5,0.3)	$S(B_1) = -0.4114, S(B_2) = -0.3419, S(B_3) = -0.2852, S(B_4) = -0.2008$	$B_4 > B_3 > B_2 > B_1$	B_4
ψ = (0.5,0.7,0.3)	$S(B_1) = -0.4519, S(B_2) = -0.4088, S(B_3) = -0.3434, S(B_4) = -0.2601$	$B_4 > B_3 > B_2 > B_1$	B_4

Fig. 1 Sensitivity results for CPFEHA operator



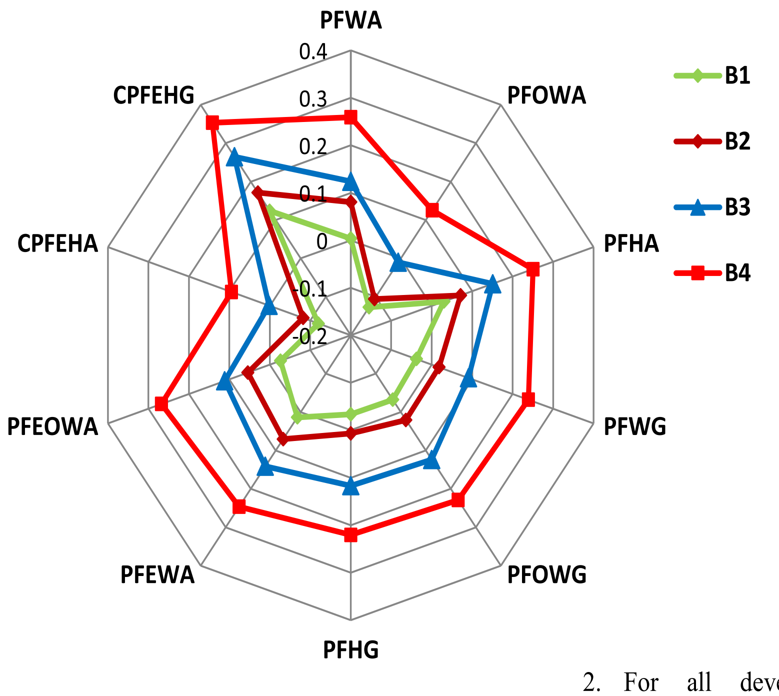
Comparative Analysis

This section includes a comparative analysis to show the stability and consistency of the novel operators. The PFWA, PFOWA, PFHA, PFWG, PFOWG, PFHG, PFEWA, and PFEOWA operators are used to compare the results (Wei 2017b; Khan et al. 2019). Table 7 summarizes the computed results, which are also plotted in Fig. 2.

Table 7: The aggregating results by different operators

	Operator	Score values	Ranking Results	Best alternative
Existing operators	PFWA	$S(B_1) = 0.0036, S(B_2) = 0.0807,\ S(B_3) = 0.1237, S(B_4) = 0.2591$	$B_4 > B_3 > B_2 > B_1$	B_4
	PFOWA	$S(B_1) = -0.1271, S(B_2) = -0.1056, S(B_3) = -0.0101, S(B_4) = 0.1256$	$B_4 > B_3 > B_2 > B_1$	B_4
	PFHA	$S(B_1) = 0.0300, S(B_2) = 0.0712, S(B_3) = 0.1508, S(B_4) = 0.2504$	$B_4 > B_3 > B_2 > B_1$	B_4
	PFWG	$S(B_1) = -0.0376, S(B_2) = 0.0182, S(B_3) = 0.0909, S(B_4) = 0.2387$	$B_4 > B_3 > B_2 > B_1$	B_4
	PFOWG	$S(B_1) = -0.0323, S(B_2) = 0.0206, S(B_3) = 0.1240, S(B_4) = 0.2288$	$B_4 > B_3 > B_2 > B_1$	B_4
	PFHG	$S(B_1) = -0.0338, S(B_2) = 0.0070, S(B_3) = 0.1170, S(B_4) = 0.2204$	$B_4 > B_3 > B_2 > B_1$	B_4
	PFEWA	$S(B_1) = 0.0138, S(B_2) = 0.0703, S(B_3) = 0.1414, S(B_4) = 0.2460$	$B_4 > B_3 > B_2 > B_1$	B_4
	PFEOWA	$S(B_1) = -0.0261, S(B_2) = 0.0543, S(B_3) = 0.1114, S(B_4) = 0.2678$	$B_4 > B_3 > B_2 > B_1$	B_4
Proposed operators	CPFEHA	$S(B_1) = -0.1201, S(B_2) = -0.0821, S(B_3) = 0.0022, S(B_4) = 0.0950$	$B_4 > B_3 > B_2 > B_1$	B_4
	CPFEHG	$S(B_1) = 0.1241, S(B_2) = 0.1714, S(B_3) = 0.2646, S(B_4) = 0.3535$	$B_4 > B_3 > B_2 > B_1$	B_4

Fig.2 Radar graph for comparison in which scale of grid representing score values



The following observations have been made based on the comparative analysis.

1. Table 7 and Fig. 2 clearly show that all developed and existing aggregating operators came to the same conclusion, namely that B_4 and B_1 are the best and worst, respectively.
2. For all developed and existing aggregating operators, the ranking order remains same.
3. The CPFEHA and CPFEHG operators are converted into existing PFEHA and PFEHG operators when confidence level $\psi = 1$.

In comparison to some existing aggregating operators, the proposed aggregating operators are more general, flexible, stable, and consistent, and provide more realistic results to handle MCGDM problems in a PFS environment.

Conclusion

The paper proposed a novel MCGDM problem in a PFS Environment by incorporating an expert's familiarity degree through a confidence level and using Einstein operations. CPFEHA and CPFEHG are two new aggregating operators developed in this paper. The proposed aggregation operators take into account not only the decision makers' evaluation information in terms of PFN, but also their levels of familiarity with the problem in terms of confidence level. Also discussed are some desirable properties and special cases for proposed aggregating operators. Then, using novel aggregating operators, a MCGDM problem of air quality evaluation was investigated. Finally, sensitivity and comparative analyses were performed to assess the validity and effectiveness of the proposed aggregation operators. The following are the most notable points:

1. All novel and existing aggregating operators reach the same conclusion, namely that alternatives and are best and worst for the problem under consideration, respectively.
2. Novel aggregating operators preserve the nature (increasing or decreasing) of relative score values.
3. The proposed novel aggregating operators are more general, flexible, stable, and consistent, resulting in more realistic and reliable results.

This study can be extended for different operational laws, aggregation operators, fuzzy environment, etc.

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