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Compounded Lomax Distribution: Properties And Application

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Abstract : In present study, we establish a new compounded model by combining two cumulative density functions (cdf's) of Lomax and Rayleigh distribution which provides a more suitable model for representing lifetime data.We study the statistical and its algebraicproperties of the derived distribution. We calculate the expressions for itsdistribution function, survival function, hazard rate function, order statistics, Renyi entropy, moment generating function and wth moment.Graphs are drawnfor this distributionto study the effect of parameters. We discuss estimation using technique of maximum likelihood estimation and discuss its application over some existing probability distributions.

Keywords:: Rayleigh distribution;Maximum likelihood estimation;Lomax distribution; Moment generating function;Moments;Order Statistics.

AMS Subject Code: 62F10, 62F03.

Introduction

To model circumstances in fields such as medical sciences, actuarial, environmental sciences, engineering, biological studies, economics and demographics many probability distributions are defined. Some extend families of well known distributions are introduced providing greater flexibility to model data. However, there still remain gaps where the real dataset does not fits to particular probability models. Determining an adequate model to make inferences is a very important problem.

The initiative of developing new distributions remains an important topic in the modern literatures. Recent literature focus on new techniques for introducing significant distributions. Marshall and Olkin [5] , Azzalini [2] and Gupta et al. [17] are the worth mentioning in this area. Eugene et al. [15] proposed beta-generated method. A new transformation method by adding one parameter also generated by Shaw and Buckley [18]. An extended-G geometric family generated due to Cordeiro, Silva and Ortega [7]. Cordeiro et al. [6] proposed Exponentiated Weibull-H distribution Family. Power Lindley-G Family of distributions is obtained by Hassan and Nassr [4]. A new family of Distribution by Modi et al. [11]. Simulation based estimation study by Modi et al.[12].

Here, we introduce a new approach of combining the cdf's of two known distributions to define the Compounded Lomax distribution. Remaining paper is divided as follows. Some basic definitions are given which will be required in subsequent sections. In Subsection 2.4, we derive the density function of Compounded Lomax distribution and plot graphs of its density function and distribution function. We also obtain the formulae for moment generating function, hazard rate function, moments and the survival function for this distribution in subsequent Subsections. In Subsection 2.8 and 2.9, we calculate Renyi entropy and order statistic distribution for proposed model respectively. In Subsection 2.10, we estimate parameters by using the technique of maximum-likelihood estimation. In Section 3, we provide application of derived distribution over existing ones.

Lemma 1 From Integrals and Series [8], Equation (3.326.2), Page 337.

For $Re\,\beta > 0, Re\,m > 0, Re\,n > 0,$ we have

$$\int_0^\infty x^m \exp\left(-\beta . x^n\right) dx = \frac{\Gamma \nu}{n.\beta^\nu}, \qquad \nu = \frac{m+1}{n}$$

Lemma 2 From Integrals and Series [8], Equation (3.462.5), Page 365.

For $Re\,\tau > 0, |\arg \nu| < \frac{\pi}{2}$, we have

$$\int_0^\infty x.\exp\left\{-\tau x^2 - 2\nu x\right\} dx = \frac{1}{2\tau} - \frac{\nu}{2\tau}\sqrt{\frac{\pi}{\tau}}.e^{\nu^2/\tau}\left\{1 - erf\left(\frac{\nu}{\sqrt{\tau}}\right)\right\}$$

Lemma 3 From Series and Integrals[3], Volume 1, Equation (5.2.11.3), Page 712.

$$\frac{m!}{(1-x)^{m+1}} = \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} x^k$$

Lemma 4 From Integrals and Series [8], Equation (1.110), Page 25.

If δ is a positive integer and |x|≤1, then:

$$(1-x)^{\delta-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\delta-1}{j} x^j$$

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Materials and methods

"*Lomax Distribution*"

The Lomax distribution considered by Lomax [12] and Moghadam et al. [14] is widely used in actuarial modelling , economics and business. The pdf of Lomax distribution is defined as

$$f(x;l,\beta)=\frac{l}{\beta}\left(1+\frac{x}{\beta}\right)^{-(1+l)} \quad l>0,$$

β>0, x≥0 (1)

It's cdf is given by

$$F(x;l,\beta)=1-\left(1+\frac{x}{\beta}\right)^{-l}$$

l>0,β>0,x≥0 (2)

"*Rayleigh Distribution*"

Rayleigh distribution is derived by Lord Rayleigh [13] which is widely used in communication engineering, applied statistics and reliability analysis. Thus a random variable Y has Rayleigh distribution, if its pdf g(y) and cdf G(y) are, respectively, given by:

$$g(y)=\frac{y}{a^2}.e^{-y^2/2a^2}$$

y>0,a>0 (3)

and

$$G(y)=1-e^{-y^2/2a^2} \quad (4)$$

"*The distribution using composition of cdf's*"

Let F(x) be the cdf and f(x) be density function of a statistical distribution referred as the first. Similarly cdf B(x) and b(x) be the pdf of the second distribution. Then the new distribution's cdf is given by

$$F_B(x)=\frac{F\left(B(x)\right)}{F(1)} \quad (5)$$

$$f_B(x)=\frac{F'\left(B(x)\right)}{F(1)}b(x) \quad (6)$$

Clearly the support of the defined distribution is same as that of the second distribution. Thus for different combination of first and second distributions we can establish a variety of modified distributions. The additional advantage of the modified distribution is that it is more adjustable with improved number of parameters.

"*Derivation of cdf and pdf of Compounded Lomax distribution*:"

For the distribution function given in equation (5), using equation (2) and (4), we obtain the distribution function for Compounded Lomax distribution as:

$$F_X(x)=\frac{1-\left[1+\frac{1-e^{-x^2/2a^2}}{\beta}\right]^{-l}}{1-\left[1+\frac{1}{\beta}\right]^{-l}} =\frac{\beta^{-l}-\left(1+\beta-e^{-x^2/2a^2}\right)^{-l}}{\beta^{-l}-\left(1+\beta\right)^{-l}} \quad (7)$$

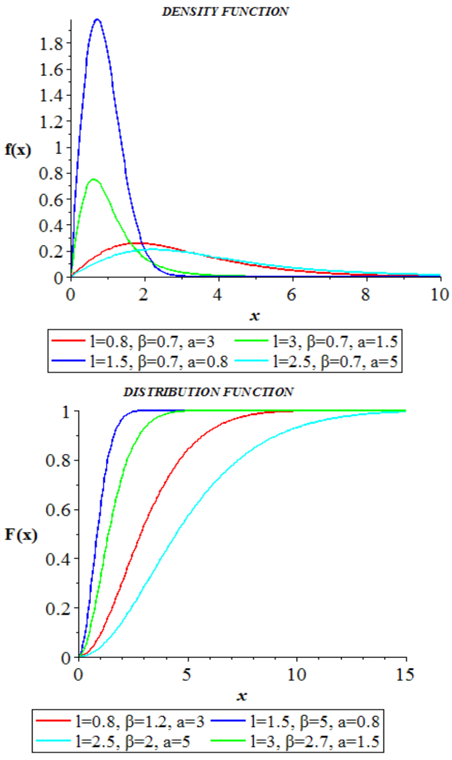
In equation (6), using equations (2) , (3) and (4), we obtain the density function for Compounded Lomax distribution as:

$$f_X(x)=\frac{l(\beta+1)^{-(l+1)}}{\left[\beta^{-l}-(1+\beta)^{-l}\right].a^2}\left[1-\frac{e^{-x^2/2a^2}}{\beta+1}\right]^{-(l+1)} x.e^{-x^2/2a^2}$$

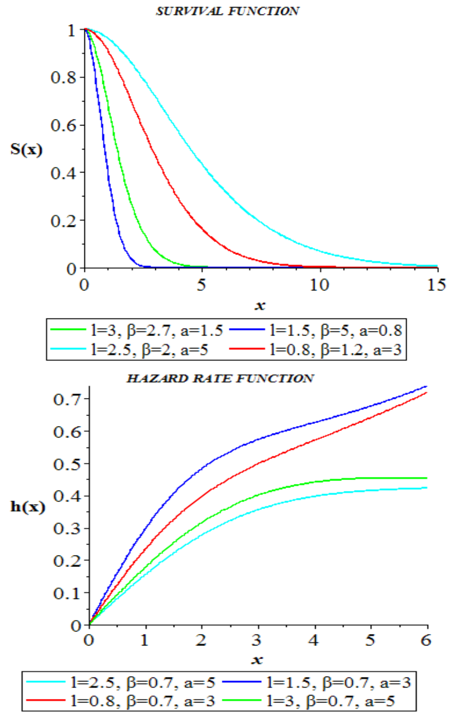
Using Lemma 3, we get

$$f_X(x)=\frac{l(\beta+1)^{-(l+1)}}{\left[\beta^{-l}-(1+\beta)^{-l}\right].a^2}\sum_{k=0}^{\infty}\frac{(l+1)_k}{k!(\beta+1)^k}.x.e^{-x^2(k+1)/2a^2}$$

l, β, a>0,x≥0 (8)



Graph of density and distribution function of Compounded Lomax distribution for some set of parameters l, β and a.



Graph of hazard rate function and survival function of Compounded Lomax distribution for some set of parameters l, β and a.

"Hazard Rate Function And Survival Function"

The survival and hazard rate function for defined density in equation (8) can be given as:

$$h_R(x) = \frac{f_R(x)}{1 - F_R(x)}$$

$$= \frac{lx.\left[\beta + 1 - e^{-x^2/2a^2}\right]^{-(l+1)} e^{-x^2/2a^2}}{a^2\left[\left(\beta + 1 - e^{-x^2/2a^2}\right)^{-l} - (\beta+1)^{-l}\right]} \quad (9)$$

$$S_R(x) = 1 - F_R(x) = 1 - \frac{\beta^{-l} - \left[\beta + 1 - e^{-x^2/2a^2}\right]^{-l}}{\beta^{-l} - [1+\beta]^{-l}}$$

$$= \frac{-[\beta+1]^{-l} + \left[\beta + 1 - e^{-x^2/2a^2}\right]^{-l}}{\beta^{-l} - [\beta+1]^{-l}} \quad (10)$$

"Moments"

The w^{th} moment for the pdf given in equation (8) can be given as:

$$\mu_w' = E\left(x^w\right) = \int_0^\infty x^w f_X(x)\,dx$$

$$= \frac{l(\beta+1)^{-(l+1)}}{\left[\beta^{-l} - (1+\beta)^{-l}\right].a^2} \sum_{k=0}^{\infty} \frac{(l+1)_k}{k!(\beta+1)^k} . \int_0^\infty x^{w+1} e^{-x^2(k+1)/2a^2} dx$$

Using Lemma 1, we get

$$\mu_w' = \frac{l(\beta+1)^{-(l+1)}}{\left[\beta^{-l} - (1+\beta)^{-l}\right].a^2} \sum_{k=0}^{\infty} \frac{(l+1)_k\, 2^{w/2} a^{(r+2)} \Gamma\left(\frac{w}{2}+1\right)}{k!(\beta+1)^k (k+1)^{\left(\frac{w}{2}+1\right)}}$$

(11)

"Moments Generating Function"

The moment generating function for the pdf defined in equation (8) can be given as:

$$M_X(q) = E(e^{q.x}) = \int_0^\infty e^{q.x} f_X(x)\,dx$$

$$=\frac{l\left(\beta+1\right)^{-(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right].a^2}\sum_{h=0}^{\infty}\frac{\left(l+1\right)_h}{h!\left(\beta+1\right)^h}.\int_0^{\infty}x.e^{-x^2(h+1)/2a^2}e^{qx}dx$$

$$=\frac{l\left(\beta+1\right)^{-(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right].a^2}\sum_{h=0}^{\infty}\frac{\left(l+1\right)_h}{h!\left(\beta+1\right)^h}.\int_0^{\infty}x.\exp\left\{-\frac{x^2\left(h+1\right)}{2a^2}+qx\right\}dx$$

Using Lemma 2, we get

$$M_X(q)=\frac{l\left(\beta+1\right)^{-(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right]}\sum_{h=0}^{\infty}\frac{\left(l+1\right)_h}{h!\left(\beta+1\right)^h}.\frac{1}{\left(h+1\right)}$$
$$\left[1+\frac{qa}{2}\sqrt{\frac{2\pi}{\left(h+1\right)}}.e^{t^2a^2/2(h+1)}\left\{1+erf\left(\frac{qa}{\sqrt{2\left(h+1\right)}}\right)\right\}\right] \tag{12}$$

"*Renyi Entropy*"

The entropy of a random variable provides the measurement of variation of the uncertainty. This entropy was given by Renyi [1] as:

$$E_X(\eta)=\frac{1}{(1-\eta)}\ln\left(\int_{-\infty}^{\infty}(f_X(x))^{\eta}dx\right),$$
$$\eta>0,\eta\neq 1$$

Using density function defined in equation (8), we get

$$E_X(\eta)=\frac{1}{(1-\eta)}\ln\left(\frac{l^{\eta}\left(\beta+1\right)^{-\eta(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right]^{\eta}a^{2\eta}}\int_0^{\infty}\left(\left[1-\frac{e^{-x^2/2a^2}}{\beta+1}\right]^{-(l+1)}x.e^{-x^2/2a^2}\right)^{\eta}dx\right)$$

Using lemma 3, we get

$$E_X(\eta)=\frac{1}{(1-\eta)}\ln\left(\frac{l^{\eta}\left(\beta+1\right)^{-\eta(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right]^{\eta}a^{2\eta}}\int_0^{\infty}\frac{x^{\eta}.e^{-\eta x^2/2a^2}}{\left(\eta\left(l+1\right)-1\right)!}\sum_{s=0}^{\infty}\frac{\left(\eta\left(l+1\right)-1+s\right)!e^{-sx^2/2a^2}}{s!.\left(\beta+1\right)^s}dx\right)$$

$$E_X(\eta)=\frac{1}{(1-\eta)}\ln\left(\frac{l^{\eta}\left(\beta+1\right)^{-\eta(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right]^{\eta}a^{2\eta}}\sum_{s=0}^{\infty}\frac{\left(\eta\left(l+1\right)-1+s\right)!}{s!.\left(\beta+1\right)^s\left(\eta\left(l+1\right)-1\right)!}\int_0^{\infty}x^{\eta}.e^{-\left(\frac{(\eta+s)x^2}{2a^2}\right)}dx\right)$$

$$E_X(\eta)=\frac{1}{(1-\eta)}\ln\left(\frac{l^{\eta}\left(\beta+1\right)^{-\eta(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right]^{\eta}a^{2\eta}}\sum_{s=0}^{\infty}\frac{\left(\eta\left(l+1\right)-1+s\right)!}{s!.2\left(\beta+1\right)^s\left(\eta\left(l+1\right)-1\right)!}\frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\left(\frac{\eta+s}{2a^2}\right)^{\left(\frac{\eta+1}{2}\right)}}\right) \tag{13}$$

"*Order Statistics*"

Here we calculate formulae for the density of the r^{th} order statistic of the Compounded Lomax distribution (CLD). Let $X_1,............,X_{q-1},X_q$ denotes a sample from Compounded Lomax distribution. Assume $X_{(1:q)}\leq............\leq X_{(q-1:q)}\leq X_{(q:q)}$ denote the order statistics obtained from above sample. The probability density function $f_{r:q}(x)$ of the r[th] order statistics $X_{r:q}$, r=1, 2 … q, given by :

$$f_{r:q}(x)=C_{r:q}\left[1-F(x;l,a,\beta)\right]^{q-r}\left[F(x;l,a,\beta)\right]^{r-1}f(x;l,a,\beta) \tag{14}$$

here we use $f\left(.\right)$ and $F\left(.\right)$ of new distribution

$$C_{r:q}=\frac{q!}{(q-r)!(r-1)!}$$

$$f_{r:q}(x)=C_{r:q}\left[\frac{\beta^{-l}-\left(1+\beta-e^{-x^2/2a^2}\right)^{-l}}{\beta^{-l}-\left(1+\beta\right)^{-l}}\right]^{r-1}\left[\frac{-\left(1+\beta\right)^{-l}+\left(1+\beta-e^{-x^2/2a^2}\right)^{-l}}{\beta^{-l}-\left(1+\beta\right)^{-l}}\right]^{q-r}$$
$$\frac{l\left(1+\beta\right)^{-(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right].a^2}\left[1-\frac{e^{-x^2/2a^2}}{1+\beta}\right]^{-(l+1)}x.e^{-x^2/2a^2}$$

$$f_{r:q}(x)=\frac{C_{r:q}lx.e^{-x^2/2a^2}\left(\beta+1-e^{-x^2/2a^2}\right)^{-(l+1)}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right]^q.a^2}\left[\beta^{-l}-\left(\beta+1-e^{-x^2/2a^2}\right)^{-l}\right]^{r-1}$$
$$\left[\left(\beta+1-e^{-x^2/2a^2}\right)^{-l}-\left(1+\beta\right)^{-l}\right]^{q-r}$$

Using Lemma 4, we get

$$f_{r:q}(x)=\frac{l.C_{r:q}}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right]^q.a^2}\sum_{k=0}^{r-1}\sum_{i=0}^{q-r}(-1)^{q+k-r-i}\binom{r-1}{k}\binom{q-r}{i}$$
$$\beta^{-l(r-1-k)}\left(\beta+1\right)^{-l(q-r-i)}\left(\beta+1-e^{-x^2/2a^2}\right)^{-(l+1+li+lk)}x.e^{-x^2/2a^2} \tag{15}$$

"*Maximum Likelihood Estimators*"

The pdf of proposed distribution is defined as:

$$f_X\left(x\right)=\frac{l}{\left[\beta^{-l}-\left(1+\beta\right)^{-l}\right].a^2}\left(\beta+1-e^{-x^2/2a^2}\right)^{-(l+1)}x.e^{-x^2/2a^2}$$

We can write its log-likelihood function as:

$$L(l,\beta,a) = n\ln l - 2n\ln a - n\ln\left(\beta^{-l} - (\beta+1)^{-l}\right)$$
$$-\sum_{i=0}^{\infty}\ln x_i - \sum_{i=0}^{\infty}\frac{x_i^2}{2a^2} - (l+1)\sum_{i=0}^{\infty}\ln\left(\beta+1-e^{-x_i^2/2a^2}\right) \quad (16)$$

We obtain normal equations as:

$$\frac{\partial L}{\partial l} = \frac{n}{l} - \frac{n}{-(\beta+1)^{-l}+\beta^{-l}}\left[\beta^{-l}\ln\beta - (\beta+1)^{-l}\ln(\beta+1)\right] - \sum_{i=0}^{\infty}\ln\left(\beta+1-e^{-x_i^2/2a^2}\right) \quad (17)$$

$$\frac{\partial L}{\partial \beta} = -\frac{n}{-(\beta+1)^{-l}+\beta^{-l}}\left[-l\beta^{-l-1} + l(\beta+1)^{-l-1}\right] - (l+1)\sum_{i=0}^{\infty}\frac{1}{\left(\beta+1-e^{-x_i^2/2a^2}\right)} \quad (18)$$

$$\frac{\partial L}{\partial a} = -\frac{2n}{a} + \sum_{i=0}^{\infty}\frac{x_i^2}{a^3} - \frac{(l+1)}{a^3}\sum_{i=0}^{\infty}\frac{\left(x_i^2 e^{-x_i^2/2a^2}\right)}{\left(\beta+1-e^{-x_i^2/2a^2}\right)} \quad (19)$$

We can get the expected values of the unidentified parameters by solving equations (17) - (19) numerically with the help of software.

Results and Discussions:

"*Applications*:"

Here the Compounded Lomax distribution is used on two set of observations. We observe its fitting by using R software to calculated the log-likelihood value, Akaike Information Criteria (AIC) . We can calculate AIC = - $2\log_e L$ +2d, *d* is the number of parameters. We consider following distributions at LOS α=1% :

Exponentiated Lomax distribution	Exponentiated Weibull distribution
$f(x) = ab\lambda(\lambda x+1)^{-b-1}\left(1-(\lambda x+1)^{-b}\right)^{a-1}$	$f(x) = ba\lambda^b x^{b-1}.\exp\left(-(\lambda x)^b\right)\left(1-\exp\left(-(\lambda x)^b\right)\right)^{a-1}$
Power Lomax distribution	Exponentiated gamma distribution
$f(x) = ab\lambda^a x^{b-1}\left(\lambda+x^b\right)^{-a-1}$	$f(x) = \theta\tau^2 x e^{-\tau.x}\left[1-e^{-\tau.x}(\tau x+1)\right]^{\theta-1}$

H_0: Observations fit the Compounded Lomax distribution.
H1: Observations does not fit the Compounded Lomax distribution.

Data Set 1 : This data set referred from Dasgupta [16] for the 50 observations with opening of 12 mm and sheet thickness of 3.15 mm by drilling machine : 0.04, 0.02, 0.06, 0.12, 0.14, 0.08,0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12,0.18, 0.24,0.32, 0.16,0.14,0.08,0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

Table 1 The MLE and AIC value of the Compounded Lomax distribution for data set 1.

Model	Estimates	Loglikelihood	D	AIC	p-value
Compounded Lomax distribution	l=0.07404049 β=8.23796229 a=0.13078113	254.3283	0.13135	-502.6566	0.3542
Exponentiated Weibull distribution	a=3.585594 b=1.008628 λ=12.760858	51.83217	0.20149	-97.66434	0.0345
Power Lomax distribution	a=9.9077908 b=2.2154345 λ=0.2205517	55.27053	0.10897	-104.54106	0.5927
Exponentiated Lomax distribution	a=3.3356670 b=15.0472887 λ=0.8224159	51.25955	0.16701	-96.5191	0.1229

Data Set 2 : We take data set from Cooray and Ananda [9] shows the breakdown times of Kevlar 49/epoxy strands at 90% stress level pressure : 0.01, 0.20, 0.23, 1.54, 0.01, 0.02, 1.10, 0.02, 1.33, 1.34, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 4.69, 0.07, 0.07, 0.08, 0.09, 1.80, 0.09, 0.10, 0.10, 0.83, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.24, 0.24, 0.29, 0.34, 0.35, 0.80, 2.17, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 1.43, 0.54, 0.56, 2.33, 0.60, 0.72, 0.73, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 7.89, 0.79, 0.79, 0.80, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.40, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.81, 2.02, 2.05, 2.14, 3.03, 3.03, 3.34, 4.20.

Table 2 The MLE and AIC value of the Compounded Lomax distribution for data set 2.

Model	Estimate Values	Loglikelihood	D	AIC	p-value
Compounded Lomax distribution	l=8.958211 β=1.148063 a=2.083999	- 12.50628	0.21702	31.01256	0.01477
Exponentiated gamma distribution	τ =0.4189795 θ=1.1701931	-102.9125	0.083559	209.825	0.481
Power Lomax distribution	a=11.2217231 b=0.9732603 λ=10.5778152	-103.2421	0.089788	212.4842	0.3895
Exponentiated Lomax distribution	a=0.9307190 b=6.8267910 λ=0.1514169	-103.4661	0.097082	212.9322	0.297

From Table1 and Table2 , the Compounded Lomax distribution has the maximum value of log-likelihood and minimum AIC value, so it gives better fit than the Power Lomax distribution, Exponentiated Lomax distribution, Exponentiated gamma distribution and Exponentiated Weibull distribution. We cannot reject the null hypothesis as p-value> α and claim that data follows Compounded Lomax distribution.

Conclusion

In this paper, we establish the new Compounded Lomax distribution. We observe proposed distribution is unimodal and positively skewed having reverse-J shaped survival function. The formula for hazard rate function and its plots for derived distribution are also shown. We calculate formulae for its w^{th} moment and mgf. We had derived the expressions for calculating Renyi entropy and order statistics for established distribution. To estimate its parameters we used method of MLE also shown application of derived distribution on real data sets.

Importance of the work:
We observe from application that proposed Compounded Lomax distribution provides better fit than some existing probability distributions. Thus it can contribute to fitting various datasets existing in different areas of engineering, biomedical, finance etc. thus giving opportunity to study some mathematical and statistical properties of the phenomena.

References

[1] A. Renyi, *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*,1961, 547-561.

[2] A. Azzalini, *Scandinavian Journal of* Statistics, 12(1985), 171-178.

[3] A.P. Prudnikov, Y.A. Brychkov, O.I .Marichev, *Integrals and Series* ,1(1986), New York: Gordon and Breach Science Publishers.

[4] A.S. Hassan, S.G. Nassr, *Annals of Data Science,* 2018; https://doi.org/10.1007/s40745-018-0159-y.

[5] A.W. Marshall, I. Olkin, *Biometrika,* 84(1997) 641–652.

[6] G.M. Cordeiro, A.Z. Afify, H.M. Yousof, et al., *Mediterranean Journal of Mathematics*, 14(4)(2017), 155.

[7] G.M. Cordeiro, G.O. Silva, E.M.M. Ortega, *Journal of Statistical Distributions and Applications*, 3(3)(2016), 1-16.

[8] I.S. Gradshteyn, I.M. Ryzhik, *Table of integrals, series and products*, 7th edition, Academic Press, 2007.

[9] K. Cooray, M.M.A. Ananda, *Communications in Statistics - Theory and Methods* , 37(2008), 1323-1337.

[10] K. Modi, D. Kumar, Y. Singh, *Science & Technology Asia,* 25(1)(2020), 1-10.

[11] K. Modi, et al., *Journal of Interdisciplinary Mathematics*, 24(2)(2021), 245-258.

[12] K.S. Lomax, *J Amer Stat Assoc.* , 49(1954), 847-852 .

[13] L. Rayleigh, *M.Phil. Mag.* 10(1880), 73-78.

[14] M.S. Moghadam, F. Yaghmaei, M. Bahanezhad, *Applied Mathematical Sciences,* 6 (105)(2012), 5241-5251.

[15] N. Eugene, C. Lee, F. Famoye, *Communications in Statistics - Theory and Methods* , 31(2002), 497-512 .

[16] R. Dasgupta, *Sankhya*, 73(2011), 1-19.

[17] R.C. Gupta, P.L. Gupta, R.D. Gupta, *Communications in Statistics - Theory and Methods*, 27(1998), 887–904.

[18] W. Shaw, I. Buckley, *The alchemy of probability distributions: Beyond Gram Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map*. 2007; Research Report.